Summary of the Ph.D. thesis

**Algebraic and analytic methods in graph theory**

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This thesis revolves around two main topics. In the first part we consider the behaviour of roots of graph polynomials, notably the chromatic polynomial and the matching polynomial, on a sequence of graphs that is convergent in the Benjamini-Schramm sense. In the second part we suggest a possible structural characterization for positive graphs along with some partial results to support our conjecture. The two areas are connected by the use of algebraic and analytic tools in a graph-theoretic setting, especially homomorphisms and measures but also convergence, moments, quantum graphs and some spectral theory.

For a finite graph $G$, let $\text{ch}_G(q)$ denote the number of proper colorings of $G$ with $q$ colors. Discovered by Birkhoff \[9\], this is a polynomial in $q$, called the chromatic polynomial of $G$. While some of its coefficients, roots and substitutions correspond to classical graph-theoretic invariants of $G$, chromatic roots also play an important role in statistical mechanics, where they control the behaviour of the antiferromagnetic Potts model at zero temperature. In particular, physicists are interested in the so-called thermodynamic limit, where the underlying graph is a lattice with size approaching infinity.

In the last decade, convergence of graph sequences became an important concept in mathematics. Motivated by efforts to better understand the structure of the internet, social networks and other huge networks in biology, physics and industrial processes, several theories appeared. The main idea in each of the theories is that we have a very large graph that is impractical to process or even to know its edges precisely, but we can sample it in some way and then produce smaller graphs that are structurally similar to it in the sense that they give rise to similar samples. If we have a sequence of graphs whose samples approximate those of the original graph arbitrarily closely then we say that this sequence converges to the original graph.

The most prevalent theories are the one about convergence of dense graphs by Lovász and Szegedy \[38\] and the one about convergence of bounded degree graphs by Benjamini and Schramm \[8\]. The latter one is also useful as a generalization of the aforementioned thermodynamic limit. Given a sequence of finite graphs $G_n$ with bounded degree, we call it convergent in the Benjamini-Schramm sense if for every positive $R$ and finite rooted graph $\alpha$ the probability that the $R$-ball centered at a uniform random vertex of $G_n$ is isomorphic to $\alpha$ is convergent. In other words, we can not statistically distinguish $G_n$ from $G_{n'}$ for large $n$ and $n'$ by randomly sampling them with a fixed radius of sight. For instance, we can approximate the infinite lattice $\mathbb{Z}^d$ using bricks with side lengths tending to infinity.

In Chapter 2, which is joint work with Miklós Abért, we examine the behaviour of chromatic roots on a Benjamini-Schramm convergent graph sequence. We define the root measure as the uniform distribution on the roots. We show that for a convergent sequence of graphs the root measure also converges in a certain sense.

The most natural sense here would be weak convergence, meaning that the integral of any continuous function wrt. the measure is convergent. However, this does not generally hold, as evidenced by the merged sequence of paths and cycles which is still BS convergent but the corresponding measures converge to different limits. Instead we prove convergence in holomorphic moments, showing that the integral of any holomorphic function wrt. the measure is convergent. In many cases we can use a separate argument to restrict the chromatic roots to a well-behaved set where convergence in holomorphic moments does
Our main vehicle of proof is counting homomorphisms. For two finite graphs $F$ and $G$ let $\text{hom}(F, G)$ denote the number of edge-preserving mappings from $V(F)$ to $V(G)$. Moments of the chromatic root measure can be written as a linear combination of homomorphism numbers from connected graphs. For instance, the third moment equals

$$p_3(G) = \frac{1}{8} \text{hom}(\rightarrow, G) + \frac{3}{4} \text{hom}(\rightarrow\rightarrow, G) + \frac{1}{4} \text{hom}(\bigtriangledown, G) - \frac{3}{8} \text{hom}(\bigtriangledown\bigtriangledown, G) + \frac{3}{4} \text{hom}(\bigtriangleup\bigtriangleup, G) - \frac{1}{8} \text{hom}(\bigtriangleup, G).$$

Since these homomorphism numbers converge after normalization, so do the moments themselves, therefore we get the convergence of measures. It also follows that the normalized logarithm of the chromatic polynomial, called the free energy, converges to a real analytic function outside a disc, which answers a question of Borgs [10, Problem 2].

Our results have been recently extended by Csikvári and Frenkel [16] to a much broader class of polynomials, namely multiplicative graph polynomials of bounded exponential type. In addition to the chromatic polynomial this includes the Tutte polynomial, the modified matching polynomial, the adjoint polynomial and the Laplacian characteristic polynomial.

In light of this generalization we also investigate the matching measure in Chapter 3, which is joint work with Miklós Abért and Péter Csikvári. The matching polynomial of a finite graph $G$ is defined as

$$\sum_k (-1)^k m_k(G) x^{|V(G)| - 2k}$$

where $m_k(G)$ denotes the number of matchings in $G$ with exactly $k$ edges. It also relates to statistical physics, this time to the monomer-dimer model. We can follow our path from Chapter 2 by defining the matching measure as the uniform distribution on the roots of the matching polynomial, and since the Heilmann-Lieb theorem [32] constrains these roots to a compact subset of the real line, we get weak convergence from the Csikvári-Frenkel result, allowing us to automatically extend the definition to infinite vertex transitive lattices.

Alternatively, one can use spectral theory to define the matching measure directly on an infinite vertex transitive lattice $L$. A walk in $L$ is called self-avoiding if it touches every vertex of $L$ at most once. We can define the tree of self-avoiding walks starting at $v$ by connecting two of them if one is a one step extension of the other. As proved in Chapter 3, our previous definition for the matching measure of $L$ is equivalent to the spectral measure of this tree.

We continue by expressing the free energies of monomer-dimer models on Euclidean lattices from their respective matching measures, which allows us to give new, strong estimates. While free energies are traditionally estimated using the Mayer series, the advantage of our approach is that certain natural functions can be integrated along the measure even if the corresponding series do not converge.

In general, no explicit formulae are known for the matching measures themselves, only in some special cases like the infinite $d$-regular tree. We can show, however, that the matching measure of a broad class of infinite lattices is atomless.
In Chapter 4 we turn our focus towards positive graphs. This chapter is joint work with Omar Antolín Camarena, Endre Csóka, Gábor Lippner and László Lovász. We already considered homomorphism numbers in Chapter 2, but here we extend the definition to allow weighted target graphs. By adding a real weight \( w_{ij} \) to each edge \( ij \) of the finite graph \( G \) we have
\[
\text{hom}(F,G) = \sum_{\varphi : V(F) \to V(G)} \prod_{ij \in E(F)} w_{\varphi(i)\varphi(j)}.
\]
Using arbitrary real edge weights means that the homomorphism number can easily become negative. It turns out, however, that there are certain finite graphs \( F \) that always exhibit a nonnegative \( \text{hom}(F,G) \) regardless of the weighted graph \( G \). We call such an \( F \) a positive graph.

The following are some examples of positive graphs:

\[
\begin{array}{cccccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

while the ones below are not positive:

\[
\begin{array}{cccccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

For instance, \( K_3 \) is not positive, since \( \text{hom}(K_3,G) < 0 \) if \( G \) is a copy of \( K_3 \) with all edges having weight \(-1\). This construction shows that no graph having an odd number of edges can be positive.

But why is the cycle of length 4 positive? We can write
\[
\text{hom}(C_4,G) = \sum_{\varphi} w_{\varphi(1)\varphi(2)}w_{\varphi(2)\varphi(3)}w_{\varphi(3)\varphi(4)}w_{\varphi(4)\varphi(1)} = \\
= \sum_{\varphi(1)} \left( \sum_{\varphi(2)} w_{\varphi(1)\varphi(2)}w_{\varphi(2)\varphi(3)} \right) \left( \sum_{\varphi(4)} w_{\varphi(3)\varphi(4)}w_{\varphi(4)\varphi(1)} \right) = \\
= \sum_{\varphi(1)} \left( \sum_{\varphi(2)} w_{\varphi(1)\varphi(2)}w_{\varphi(2)\varphi(3)} \right)^2
\]

Once we fix the images of two opposite vertices, the number of homomorphisms into any target graph \( G \) can be written as a square. So the total homomorphism number is a sum of squares and thus nonnegative.

This construction can be generalized. Suppose we have a graph \( H \) where the vertices \( s_1, s_2, \ldots, s_k \) form an independent set. Let \( H' \) be a disjoint copy of \( H \) and identify each \( s_i \) with \( s'_i \). A graph \( F \) obtained this way is called symmetric.
Once we fix the images of the $s_i$’s, $H$ and $H'$ have the same number of homomorphisms into our target graph $G$, and these mappings are independent from each other. Therefore the total number of homomorphisms is again a sum of squares, and thus all symmetric graphs are positive.

We conjecture that this implication is in fact an equivalence, i.e. all positive graphs are also symmetric.

To prove some special cases of the conjecture, we introduce a partitioning technique that allows us to disprove the positivity of certain graphs. The idea is to restrict the set of possible images for each vertex. In a simplified explanation, we may color the vertices of both $F$ and $G$ and only consider those homomorphisms that map each vertex into one of the same color. There are colored graphs $F$ that feature a nonnegative $\text{hom}(F,G)$ into any colored and edge-weighted graph $G$, and these obviously only depend on the partition $\mathcal{N}$ of $V(F)$ corresponding to the coloring. Such an $\mathcal{N}$ is called a positive partition of $V(F)$.

(For the full analytic definition, see Chapter 4.)

Several operations on positive partitions preserve their positivity, such as merging classes together or restricting the underlying graph $F$ to the union of certain classes. We may also split a class according to the degrees of the vertices, or even the number of edges going from a given vertex into some other specific class.

Starting from the trivial partition on $F$ and successively dividing classes using these operations, we get to the walk-tree partition of $F$ where two vertices belong to the same class if and only the universal cover of $F$ as seen from these two vertices are isomorphic. Therefore any union of classes from the walk-tree partition of a positive graph is still positive, which immediately proves the conjecture for trees, and combined with a computer search also proves the conjecture for all graphs on at most 10 vertices, except one.

We end the chapter with some statements about positive graphs, including that they have a homomorphic image with at least half the original number of nodes, in which every edge has an even number of pre-images.

**Results from Chapter 2:**

For a simple graph $G$ let $\mu_G$, the chromatic measure of $G$ denote the uniform distribution on its chromatic roots. By a theorem of Sokal [46], $\mu_G$ is supported on the open disc of radius $Cd$ around 0, denoted by

$$D = B(0, Cd)$$

where $d$ is the maximal degree of $G$ and $C < 8$ is an absolute constant.
**Theorem 2.1.** Let \((G_n)\) be a Benjamini-Schramm convergent graph sequence of absolute degree bound \(d\), and \(\bar{D}\) an open neighborhood of the closed disc \(D\). Then for every holomorphic function \(f : \bar{D} \to \mathbb{C}\), the sequence

\[
\int_{\bar{D}} f(z) d\mu_{G_n}(z)
\]

converges.

Let \(\ln\) denote the principal branch of the complex logarithm function. For a simple graph \(G\) and \(z \in \mathbb{C}\) let

\[
t_G(z) = \frac{\ln \text{ch}_G(z)}{|V(G)|}
\]

where this is well-defined. In statistical mechanics, \(t_G(z)\) is called the entropy per vertex or the free energy at \(z\). In their recent paper [11], Borgs, Chayes, Kahn and Lovász proved that if \((G_n)\) is a Benjamini-Schramm convergent graph sequence of absolute degree bound \(d\), then \(t_{G_n}(q)\) converges for every positive integer \(q > 2d\). Theorem 2.1 yields the following.

**Theorem 2.2.** Let \((G_n)\) be a Benjamini-Schramm convergent graph sequence of absolute degree bound \(d\) with \(|V(G_n)| \to \infty\). Then \(t_{G_n}(z)\) converges to a real analytic function on \(\mathbb{C} \setminus D\).

In particular, \(t_{G_n}(z)\) converges for all \(z \in \mathbb{C} \setminus D\). Theorem 2.2 answers a question of Borgs [10, Problem 2] who asked under which circumstances the entropy per vertex has a limit and whether this limit is analytic in \(1/z\). Note that for an amenable quasi-transitive graph and its Følner sequences, this was shown to hold in [42].

Weak convergence of \(\mu_{G_n}\) isn’t true in general, but it holds for some natural sequences of graphs. For example, let \(T_n = C_4 \times P_n\) denote the \(4 \times n\) tube, i.e. the cartesian product of the 4-cycle with a path on \(n\) vertices. \(T_n\) is a 4-regular graph except at the ends of the tube.

**Proposition 2.3.** The chromatic measures \(\mu_{T_n}\) weakly converge.

Another naturally interesting case is when the girth of \(G\) (the minimal size of a cycle) is large. One can show that

\[
\int_{\bar{D}} z^k d\mu(z) = \frac{|E(G)|}{|V(G)|} \quad (1 \leq k \leq \text{girth}(G) - 2)
\]

that is, the moments are all the same until the girth is reached (see Lemma 2.13). This implies that for a sequence of \(d\)-regular graphs \(G_n\) with girth tending to infinity, the limit of the free entropy

\[
\lim_{n \to \infty} t_{G_n}(z) = \ln q + \frac{d}{2} \ln \left(1 - \frac{1}{q}\right)
\]

for \(q > Cd\). This is one of the main results in [6]. Note that their proof works for \(q > d + 1\), while our approach only works for \(q > Cd\). The advantage of our approach is that it gives an explicit estimate on the number of proper colorings of large girth graphs.
**Theorem 2.4.** Let $G$ be a finite graph of girth $g$ and maximal degree $d$. Then for all $q > Cd$ we have

$$\left| \frac{\ln \chi_G(q)}{|V(G)|} - \left( \ln q + \frac{|E(G)|}{|V(G)|} \ln(1 - \frac{1}{q}) \right) \right| \leq 2\left( \frac{(Cd/q)^{g-1}}{1 - Cd/q} \right).$$

**Results from Chapter 3:**

**Definition 3.1.** Let $L$ be an infinite vertex transitive lattice. The matching measure $\rho_L$ is the spectral measure of the tree of self-avoiding walks of $L$ starting at $v$, where $v$ is a vertex of $L$.

For a finite graph $G$, let $\rho_G$: the matching measure of $G$ be the uniform distribution on the roots of $\mu(G,x)$. Using previous work of Godsil [25] we show that $\rho_L$ can be obtained as the thermodynamical limit of the $\rho_{G_n}$.

**Theorem 3.2.** Let $L$ be an infinite vertex transitive lattice and let $G_n$ Benjamini–Schramm converge to $L$. Then $\rho_{G_n}$ weakly converges to $\rho_L$ and $\lim_{n \to \infty} \rho_{G_n}(\{x\}) = \rho_L(\{x\})$ for all $x \in \mathbb{R}$.

Let $G$ be a finite graph, and recall that $|G|$ denotes the number of vertices of $G$, and $m_k(G)$ denotes the number of $k$-matchings ($m_0(G) = 1$). Let $t$ be a non-negative real number, and

$$M(G,t) = \sum_{k=0}^{|G|/2} m_k(G)t^k,$$

We call $M(G,t)$ the matching generating function or the partition function of the monomer-dimer model. Clearly, it encodes the same information as the matching polynomial. Let

$$p(G,t) = \frac{2t \cdot M'(G,t)}{|G| \cdot M(G,t)},$$

and

$$F(G,t) = \frac{\ln M(G,t)}{|G|} - \frac{1}{2} p(G,t) \ln(t).$$

Note that

$$\tilde{\lambda}(G) = F(G,1)$$

is called the monomer-dimer free energy.

The function $p = p(G,t)$ is a strictly monotone increasing function which maps $[0, \infty)$ to $[0, p^*)$, where $p^* = \frac{2\nu(G)}{|G|}$, where $\nu(G)$ denotes the number of edges in the largest matching. If $G$ contains a perfect matching, then $p^* = 1$. Therefore, its inverse function $t = t(G,p)$ maps $[0, p^*)$ to $[0, \infty)$. (If $G$ is clear from the context, then we will simply write $t(p)$ instead of $t(G,p)$.) Let

$$\lambda_G(p) = F(G,t(p))$$
if $p < p^*$, and $\lambda_G(p) = 0$ if $p > p^*$.

Note that we have not defined $\lambda_G(p^*)$ yet. We simply define it as a limit:

$$\lambda_G(p^*) = \lim_{p \to p^*} \lambda_G(p).$$

We will show that this limit exists, see part (d) of Proposition 3.15. Later we will extend the definition of $p(G, t), F(G, t)$ and $\lambda_G(p)$ to infinite lattices $L$.

The intuitive meaning of $\lambda_G(p)$ is the following. Assume that we want to count the number of matchings covering $p$ fraction of the vertices. Let us assume that it makes sense: $p = \frac{2k}{|G|}$, and so we wish to count $m_k(G)$. Then

$$\lambda_G(p) \approx \frac{\ln m_k(G)}{|G|}.$$ 

For the more precise formulation of this statement, see Proposition 3.15.

Our next aim is to extend the definition of the function $\lambda_G(p)$ for infinite lattices $L$. We also show an efficient way of computing its values if $p$ is sufficiently separated from $p^*$.

**Theorem 3.16.** Let $(G_n)$ be a Benjamini–Schramm convergent sequence of bounded degree graphs. Then the sequences of functions

(a) $p(G_n, t)$,

(b) $\frac{\ln M(G_n, t)}{|G_n|}$

converge to strictly monotone increasing continuous functions on the interval $[0, \infty)$.

If, in addition, every $G_n$ has a perfect matching then the sequences of functions

(c) $t(G_n, p)$,

(d) $\lambda_{G_n}(p)$

are convergent for all $0 \leq p < 1$.

**Definition 3.18.** Let $L$ be an infinite lattice and $(G_n)$ be a sequence of finite graphs which is Benjamini–Schramm convergent to $L$. For instance, $G_n$ can be chosen to be an exhaustion of $L$. Then the sequence of measures $(\rho_{G_n})$ weakly converges to some measure which we will call $\rho_L$, the matching measure of the lattice $L$. For $t > 0$, we can introduce

$$p(L, t) = \int \frac{t z^2}{1 + tz^2} \, d\rho_L(z)$$

and

$$F(L, t) = \int \frac{1}{2} \ln (1 + t z^2) \, d\rho_L(z) - \frac{1}{2} p(L, t) \ln(t).$$
The monomer-dimer free energy of a lattice $L$ is
\[
\tilde{\lambda}(L) = F(L, 1) = \int \frac{1}{2} \ln (1 + z^2) \, d\rho_L(z)
\]

The following table contains some numerical results.

<table>
<thead>
<tr>
<th>Lattice</th>
<th>$\lambda(L)$</th>
<th>Bound on error</th>
<th>$p(L, 1)$</th>
<th>Bound on error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2d</td>
<td>0.6627989725</td>
<td>$3.72 \cdot 10^{-8}$</td>
<td>0.638123105</td>
<td>$5.34 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>3d</td>
<td>0.7859659243</td>
<td>$9.89 \cdot 10^{-7}$</td>
<td>0.684380278</td>
<td>$1.14 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>4d</td>
<td>0.8807178880</td>
<td>$5.92 \cdot 10^{-6}$</td>
<td>0.715846906</td>
<td>$5.86 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>5d</td>
<td>0.9581235802</td>
<td>$4.02 \cdot 10^{-5}$</td>
<td>0.739160383</td>
<td>$3.29 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>6d</td>
<td>1.0237319240</td>
<td>$1.24 \cdot 10^{-4}$</td>
<td>0.757362382</td>
<td>$8.91 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>7d</td>
<td>1.0807591953</td>
<td>$3.04 \cdot 10^{-4}$</td>
<td>0.772099489</td>
<td>$1.95 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>hex</td>
<td>0.5817003663</td>
<td>$1.56 \cdot 10^{-9}$</td>
<td>0.600508638</td>
<td>$2.65 \cdot 10^{-8}$</td>
</tr>
</tbody>
</table>

In general, the matching measure $\rho_L$ can contain atoms. For instance, if $G$ is a finite graph then clearly $\rho_G$ consists of atoms. On the other hand, it can be shown that for all lattices in the table above, the measure $\rho_L$ is atomless.

**Theorem 3.22.** Let $L$ be a lattice satisfying one of the following conditions.

(a) The lattice $L$ can be obtained as a Benjamini–Schramm limit of a finite graph sequence $G_n$ such that $G_n$ can be covered by $o(|G_n|)$ disjoint paths.

(b) The lattice $L$ can be obtained as a Benjamini–Schramm limit of connected vertex transitive finite graphs.

Then the matching measure $\rho_L$ is atomless.

**Results from Chapter 4:**

We call the graph $G$ positive if $\text{hom}(G, H) \geq 0$ for every edge-weighted graph $H$ (where the edgeweights may be negative). It would be interesting to characterize these graphs; in this chapter we offer a conjecture and line up supporting evidence.

We call a graph symmetric, if its vertices can be partitioned into three sets $(S, A, B)$ so that $S$ is an independent set, there is no edge between $A$ and $B$, and there exists an isomorphism between the subgraphs spanned by $S \cup A$ and $S \cup B$ which fixes $S$.

**Conjecture 4.1.** A graph $G$ is positive if and only if it is symmetric.

The “if” part of the conjecture is easy.

**Lemma 4.2.** If a graph $G$ is symmetric, then it is positive.

In the reverse direction, we only have partial results. We are going to prove that the conjecture is true for trees (Corollary 4.20) and for all graphs up to 9 nodes (see Section 4.5).

We state and prove a number of properties of positive graphs. Each of these is of course satisfied by symmetric graphs.
**Lemma 4.3.** If $G$ is positive, then $G$ has an even number of edges.

We call a homomorphism even if the preimage of each edge is has even cardinality.

**Lemma 4.4.** If $G$ is positive, then there exists an even homomorphism of $G$ into itself.

For two looped-simple graphs $G_1$ and $G_2$, we denote by $G_1 \times G_2$ their categorical product, defined by

$$V(G_1 \times G_2) = V(G_1) \times V(G_2),$$

$$E(G_1 \times G_2) = \{(i_1, j_1), (i_2, j_2) : (i_1, j_1) \in E(G_1), (i_2, j_2) \in E(G_2)\}.$$

Let $K_n^+$ denote the complete graph on the vertex set $[n]$ with loops at all vertices, where $n \geq |V(G)|$.

**Theorem 4.5.** If a graph $G$ is positive, then there exists an even homomorphism $f : G \to K_n^+ \times G$ so that $|f(V(G))| \geq \frac{1}{2}|V(G)|$.

We develop a technique to show that one can partition the vertices of a positive graph in a certain way so that subgraphs spanned by each part are also positive. The main idea is to limit, over what maps $p : V \to [0, 1]$ one has to average to check positivity. Using this idea recursively we can finally reduce to maps that take each partition to disjoint subsets of $[0, 1]$. This in turn allows us to conclude positivity of the spanned subgraphs.

To this end, first we have to introduce the notion of $\mathcal{F}$-positivity. Let $G = (V, E)$ be a simple graph. For a measurable subset $\mathcal{F} \subseteq [0, 1]^V$ and a bounded measurable weight function $\omega : [0, 1] \to (0, \infty)$, we define

$$t(G, W, \omega, \mathcal{F}) = \int_{p \in \mathcal{F}} \text{hom}(G, W, \omega, p) \, dp,$$

(1)

where the weight of a $p : V \to [0, 1]$ is

$$\text{hom}(G, W, \omega, p) = \prod_{v \in V} \omega(p(v)) \prod_{e \in E} W(p(e))$$

(2)

With the measure $\mu$ with density function $\omega$ (i.e., $\mu(X) = \int_X \omega$), we can write this as

$$t(G, W, \omega, \mathcal{F}) = \int_{\mathcal{F}} \prod_{e \in E} W(p(e)) \, d\mu^V(p).$$

(3)

We say that $G$ is $\mathcal{F}$-positive if for every kernel $W$ and function $\omega$ as above, we have $t(G, W, \omega, \mathcal{F}) \geq 0$. It is easy to see that $G$ is $[0, 1]^V$-positive if and only if it is positive.

For a partition $\mathcal{P}$ of $[0, 1]$ into a finite number of sets with positive measure and a function $\pi : V \to \mathcal{P}$, we call the box $\mathcal{F}(\pi) = \{p \in [0, 1]^V : p(v) \in \pi(v) \forall v \in V\}$ a partition-box. Equivalently, a partition-box is a product set $\prod_{v \in V} S_v$, where the sets $S_v \subseteq [0, 1]$ are measurable, and either $S_u \cap S_v = \emptyset$ or $S_u = S_v$ for all $u, v \in V$. 

10
A partition $\mathcal{N}$ of $V$ is positive if for any partition $\mathcal{P}$ as above, and any $\pi : V \to \mathcal{P}$ such that $\pi^{-1}(\mathcal{P}) = \mathcal{N}$, $G$ is $\mathcal{F}(\pi)$-positive.

The walk-tree of a rooted graph $(G, v)$ is the following infinite rooted tree $R(G, v)$: its nodes are all finite walks starting from $v$, its root is the 0-length walk, and the parent of any other walk is obtained by deleting its last node. The walk-tree partition $\mathcal{R}$ is the partition of $V$ in which two nodes $u, v \in V$ belong to the same class if and only if $R(G, u) \cong R(G, v)$.

**Proposition 4.16.** If a graph $G$ is positive, then its walk-tree partition is also positive.

**Corollary 4.17.** Let $G(V, E)$ be a positive graph, and let $S \subset V$ be the union of some classes of the walk-tree partition. Then $G[S]$ is also positive.

**Corollary 4.18.** If $G$ is positive, then for each $k$ the subgraph of $G$ spanned by all nodes with degree $k$ is positive as well.

**Corollary 4.19.** For each odd $k$ the number of nodes of $G$ with degree $k$ must be even.

**Corollary 4.20.** Conjecture 4.1 is true for trees.

We checked Conjecture 4.1 for all graphs on at most 10 vertices using the previous results and a computer program. We verified that all positive graphs are symmetric, possibly except the one below, which is not symmetric, but we could not decide whether it is positive.
Bibliography


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**Figures**

Left: chromatic roots of the 30368 cubic graphs of size 32 and girth 7
Right: possible limit points of chromatic roots of $T_n = C_4 \times P_n$ as $n \to \infty$

The pyramid graph and its trees of self-avoiding walks starting from $\odot$ and $\ominus$

An approximation for the matching measure of $\mathbb{Z}^2$

An approximation for the matching measure of $\mathbb{Z}^3$