Eötvös Loránd University
Institute of Mathematics

Summary of the Ph.D. thesis

Algebraic and analytic methods in graph theory

Tamás Hubai

Doctoral School: Mathematics
Director: Miklós Laczkovich, member of the Hungarian Academy of Sciences

Doctoral Program: Pure Mathematics
Director: András Szűcs, member of the Hungarian Academy of Sciences

Supervisor: László Lovász
member of the Hungarian Academy of Sciences

Department of Computer Science, Eötvös Loránd University

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This thesis revolves around two main topics. In the first part we consider the behaviour of roots of graph polynomials, notably the chromatic polynomial and the matching polynomial, on a sequence of graphs that is convergent in the Benjamini-Schramm sense. In the second part we suggest a possible structural characterization for positive graphs along with some partial results to support our conjecture. The two areas are connected by the use of algebraic and analytic tools in a graph-theoretic setting, especially homomorphisms and measures but also convergence, moments, quantum graphs and some spectral theory.

For a finite graph G, let $\mathrm{ch}_G(q)$ denote the number of proper colorings of G with q colors. Discovered by Birkhoff [9], this is a polynomial in q, called the *chromatic polynomial* of G. While some of its coefficients, roots and substitutions correspond to classical graph-theoretic invariants of G, chromatic roots also play an important role in statistical mechanics, where they control the behaviour of the antiferromagnetic Potts model at zero temperature. In particular, physicists are interested in the so-called thermodynamic limit, where the underlying graph is a lattice with size approaching infinity.

In the last decade, convergence of graph sequences became an important concept in mathematics. Motivated by efforts to better understand the structure of the internet, social networks and other huge networks in biology, physics and industrial processes, several theories appeared. The main idea in each of the theories is that we have a very large graph that is impractical to process or even to know its edges precisely, but we can sample it in some way and then produce smaller graphs that are structurally similar to it in the sense that they give rise to similar samples. If we have a sequence of graphs whose samples approximate those of the original graph arbitrarily closely then we say that this sequence converges to the original graph.

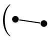
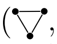
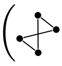
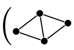
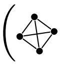
The most prevalent theories are the one about convergence of dense graphs by Lovász and Szegedy [38] and the one about convergence of bounded degree graphs by Benjamini and Schramm [8]. The latter one is also useful as a generalization of the aforementioned thermodynamic limit. Given a sequence of finite graphs G_n with bounded degree, we call it *convergent in the Benjamini-Schramm sense* if for every positive R and finite rooted graph α the probability that the R-ball centered at a uniform random vertex of G_n is isomorphic to α is convergent. In other words, we can not statistically distinguish G_n from $G_{n'}$ for large n and n' by randomly sampling them with a fixed radius of sight. For instance, we can approximate the infinite lattice $\mathbb{Z}^d$ using bricks with side lengths tending to infinity.

In Chapter 2, which is joint work with Miklós Abért, we examine the behaviour of chromatic roots on a Benjamini-Schramm convergent graph sequence. We define the *root measure* as the uniform distribution on the roots. We show that for a convergent sequence of graphs the root measure also converges in a certain sense.

The most natural sense here would be weak convergence, meaning that the integral of any continuous function wrt. the measure is convergent. However, this does not generally hold, as evidenced by the merged sequence of paths and cycles which is still BS convergent but the corresponding measures converge to different limits. Instead we prove *convergence in holomorphic moments*, showing that the integral of any holomorphic function wrt. the measure is convergent. In many cases we can use a separate argument to restrict the chromatic roots to a well-behaved set where convergence in holomorphic moments does

imply weak convergence.

Our main vehicle of proof is counting homomorphisms. For two finite graphs F and G let $\hom(F, G)$ denote the number of edge-preserving mappings from $V(F)$ to $V(G)$. Moments of the chromatic root measure can be written as a linear combination of homomorphism numbers from connected graphs. For instance, the third moment equals

$p_3(G) = \frac{1}{8}\hom($  $, G) + \frac{3}{4}\hom($  $, G) + \frac{1}{4}\hom($  $, G) -$
$\frac{3}{8}\hom\Big($  $, G\Big) + \frac{3}{4}\hom($  $, G) - \frac{1}{8}\hom\Big($  $, G\Big).$

Since these homomorphism numbers converge after normalization, so do the moments themselves, therefore we get the convergence of measures. It also follows that the normalized logarithm of the chromatic polynomial, called the free energy, converges to a real analytic function outside a disc, which answers a question of Borgs [10, Problem 2].

Our results have been recently extended by Csikvári and Frenkel [16] to a much broader class of polynomials, namely multiplicative graph polynomials of bounded exponential type. In addition to the chromatic polynomial this includes the Tutte polynomial, the modified matching polynomial, the adjoint polynomial and the Laplacian characteristic polynomial.

In light of this generalization we also investigate the matching measure in Chapter 3, which is joint work with Miklós Abért and Péter Csikvári. The *matching polynomial* of a finite graph G is defined as

$$\sum_k (-1)^k m_k(G) x^{|V(G)|-2k}$$

where $m_k(G)$ denotes the number of matchings in G with exactly k edges. It also relates to statistical physics, this time to the monomer-dimer model. We can follow our path from Chapter 2 by defining the matching measure as the uniform distribution on the roots of the matching polynomial, and since the Heilmann-Lieb theorem [32] constrains these roots to a compact subset of the real line, we get weak convergence from the Csikvári-Frenkel result, allowing us to automatically extend the definition to infinite vertex transitive lattices.

Alternatively, one can use spectral theory to define the matching measure directly on an infinite vertex transitive lattice L. A walk in L is called self-avoiding if it touches every vertex of L at most once. We can define the *tree of self-avoiding walks* starting at v by connecting two of them if one is a one step extension of the other. As proved in Chapter 3, our previous definition for the matching measure of L is equivalent to the spectral measure of this tree.

We continue by expressing the free energies of monomer-dimer models on Euclidean lattices from their respective matching measures, which allows us to give new, strong estimates. While free energies are traditionally estimated using the Mayer series, the advantage of our approach is that certain natural functions can be integrated along the measure even if the corresponding series do not converge.

In general, no explicit formulae are known for the matching measures themselves, only in some special cases like the infinite d-regular tree. We can show, however, that the matching measure of a broad class of infinite lattices is atomless.

In Chapter 4 we turn our focus towards positive graphs. This chapter is joint work with Omar Antolín Camarena, Endre Csóka, Gábor Lippner and László Lovász. We already considered homomorphism numbers in Chapter 2, but here we extend the definition to allow weighted target graphs. By adding a real weight w_{ij} to each edge ij of the finite graph G we have

$$\hom(F,G) = \sum_{\varphi:V(F)\to V(G)} \prod_{ij\in E(F)} w_{\varphi(i)\varphi(j)}.$$

Using arbitrary real edge weights means that the homomorphism number can easily become negative. It turns out, however, that there are certain finite graphs F that always exhibit a nonnegative $\hom(F,G)$ regardless of the weighted graph G. We call such an F a *positive graph.*

The following are some examples of positive graphs:

while the ones below are not positive:

For instance, K_3 is not positive, since $\hom(K_3,G) < 0$ if G is a copy of K_3 with all edges having weight -1. This construction shows that no graph having an odd number of edges can be positive.

But why is the cycle of length 4 positive? We can write

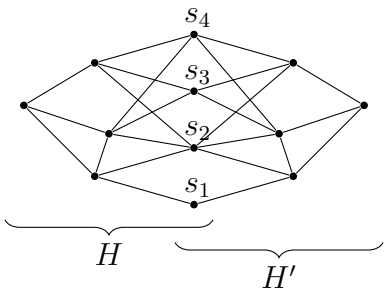
$$\hom(C_4,G) = \sum_{\varphi} w_{\varphi(1)\varphi(2)} w_{\varphi(2)\varphi(3)} w_{\varphi(3)\varphi(4)} w_{\varphi(4)\varphi(1)} =$$

$$= \sum_{\substack{\varphi(1)\\ \varphi(3)}} \Big(\sum_{\varphi(2)} w_{\varphi(1)\varphi(2)} w_{\varphi(2)\varphi(3)} \Big) \Big(\sum_{\varphi(4)} w_{\varphi(3)\varphi(4)} w_{\varphi(4)\varphi(1)} \Big) =$$

$$= \sum_{\substack{\varphi(1)\\ \varphi(3)}} \Big(\sum_{\varphi(2)} w_{\varphi(1)\varphi(2)} w_{\varphi(2)\varphi(3)} \Big)^2$$

Once we fix the images of two opposite vertices, the number of homomorphisms into any target graph G can be written as a square. So the total homomorphism number is a sum of squares and thus nonnegative.

This construction can be generalized. Suppose we have a graph H where the vertices $s_1, s_2, \dots s_k$ form an independent set. Let H' be a disjoint copy of H and identify each s_i with s_i'. A graph F obtained this way is called *symmetric.*



Once we fix the images of the s_i's, H and H' have the same number of homomorphisms into our target graph G, and these mappings are independent from each other. Therefore the total number of homomorphisms is again a sum of squares, and thus all symmetric graphs are positive.

We conjecture that this implication is in fact an equivalence, i.e. all positive graphs are also symmetric.

To prove some special cases of the conjecture, we introduce a partitioning technique that allows us to disprove the positivity of certain graphs. The idea is to restrict the set of possible images for each vertex. In a simplified explanation, we may color the vertices of both F and G and only consider those homomorphisms that map each vertex into one of the same color. There are colored graphs F that feature a nonnegative $\hom(F, G)$ into any colored and edge-weighted graph G, and these obviously only depend on the partition $\mathcal{N}$ of $V(F)$ corresponding to the coloring. Such an $\mathcal{N}$ is called a *positive partition* of $V(F)$. (For the full analytic definition, see Chapter 4.)

Several operations on positive partitions preserve their positivity, such as merging classes together or restricting the underlying graph F to the union of certain classes. We may also split a class according to the degrees of the vertices, or even the number of edges going from a given vertex into some other specific class.

Starting from the trivial partition on F and successively dividing classes using these operations, we get to the *walk-tree partition* of F where two vertices belong to the same class if and only the universal cover of F as seen from these two vertices are isomorphic. Therefore any union of classes from the walk-tree partition of a positive graph is still positive, which immediately proves the conjecture for trees, and combined with a computer search also proves the conjecture for all graphs on at most 10 vertices, except one.

We end the chapter with some statements about positive graphs, including that they have a homomorphic image with at least half the original number of nodes, in which every edge has an even number of pre-images.

Results from Chapter 2:

For a simple graph G let μ_G, the *chromatic measure of* G denote the uniform distribution on its chromatic roots. By a theorem of Sokal [46], μ_G is supported on the open disc of radius Cd around 0, denoted by

$$D = B(0, Cd)$$

where d is the maximal degree of G and $C < 8$ is an absolute constant.

Theorem 2.1. *Let (G_n) be a Benjamini-Schramm convergent graph sequence of absolute degree bound d, and $\widetilde{D}$ an open neighborhood of the closed disc $\overline{D}$. Then for every holomorphic function $f : \widetilde{D} \to \mathbb{C}$, the sequence*

$$\int_D f(z) d\mu_{G_n}(z)$$

converges.

Let ln denote the principal branch of the complex logarithm function. For a simple graph G and $z \in \mathbb{C}$ let

$$\mathrm{t}_G(z) = \frac{\ln \mathrm{ch}_G(z)}{|V(G)|}$$

where this is well-defined. In statistical mechanics, $\mathrm{t}_G(z)$ is called the *entropy per vertex* or the *free energy* at z. In their recent paper [11], Borgs, Chayes, Kahn and Lovász proved that if (G_n) is a Benjamini-Schramm convergent graph sequence of absolute degree bound d, then $\mathrm{t}_{G_n}(q)$ converges for every positive integer $q > 2d$. Theorem 2.1 yields the following.

Theorem 2.2. *Let (G_n) be a Benjamini-Schramm convergent graph sequence of absolute degree bound d with $|V(G_n)| \to \infty$. Then $\mathrm{t}_{G_n}(z)$ converges to a real analytic function on $\mathbb{C} \setminus \overline{D}$.*

In particular, $\mathrm{t}_{G_n}(z)$ converges for all $z \in \mathbb{C} \setminus \overline{D}$. Theorem 2.2 answers a question of Borgs [10, Problem 2] who asked under which circumstances the entropy per vertex has a limit and whether this limit is analytic in $1/z$. Note that for an amenable quasi-transitive graph and its Følner sequences, this was shown to hold in [42].

Weak convergence of μ_{G_n} isn't true in general, but it holds for some natural sequences of graphs. For example, let $T_n = C_4 \times P_n$ denote the $4 \times n$ tube, i.e. the cartesian product of the 4-cycle with a path on n vertices. T_n is a 4-regular graph except at the ends of the tube.

Proposition 2.3. *The chromatic measures μ_{T_n} weakly converge.*

Another naturally interesting case is when the girth of G (the minimal size of a cycle) is large. One can show that

$$\int_D z^k d\mu(z) = \frac{|E(G)|}{|V(G)|} \quad (1 \leq k \leq \mathrm{girth}(G) - 2)$$

that is, the moments are all the same until the girth is reached (see Lemma 2.13). This implies that for a sequence of d-regular graphs G_n with girth tending to infinity, the limit of the free entropy

$$\lim_{n\to\infty} \mathrm{t}_{G_n}(z) = \ln q + \frac{d}{2} \ln(1 - \frac{1}{q})$$

for $q > Cd$. This is one of the main results in [6]. Note that their proof works for $q > d+1$, while our approach only works for $q > Cd$. The advantage of our approach is that it gives an explicit estimate on the number of proper colorings of large girth graphs.

Theorem 2.4. *Let G be a finite graph of girth g and maximal degree d. Then for all $q > Cd$ we have*

$$\left|\frac{\ln \mathrm{ch}_G(q)}{|V(G)|} - \left(\ln q + \frac{|E(G)|}{|V(G)|}\ln(1-\frac{1}{q})\right)\right| \leq 2\frac{(Cd/q)^{g-1}}{1-Cd/q}.$$

Results from Chapter 3:

Definition 3.1. *Let L be an infinite vertex transitive lattice. The* matching measure ρ_L *is the spectral measure of the tree of self-avoiding walks of L starting at v, where v is a vertex of L.*

For a finite graph G, let ρ_G, the *matching measure of* G be the uniform distribution on the roots of $\mu(G,x)$. Using previous work of Godsil [25] we show that ρ_L can be obtained as the thermodynamical limit of the ρ_{G_n}.

Theorem 3.2. *Let L be an infinite vertex transitive lattice and let G_n Benjamini–Schramm converge to L. Then ρ_{G_n} weakly converges to ρ_L and $\lim_{n\to\infty}\rho_{G_n}(\{x\}) = \rho_L(\{x\})$ for all $x \in \mathbb{R}$.*

Let G be a finite graph, and recall that $|G|$ denotes the number of vertices of G, and $m_k(G)$ denotes the number of k-matchings ($m_0(G) = 1$). Let t be a non-negative real number, and

$$M(G,t) = \sum_{k=0}^{\lfloor |G|/2 \rfloor} m_k(G)t^k,$$

We call $M(G,t)$ the matching generating function or the partition function of the monomer-dimer model. Clearly, it encodes the same information as the matching polynomial. Let

$$p(G,t) = \frac{2t \cdot M'(G,t)}{|G| \cdot M(G,t)},$$

and

$$F(G,t) = \frac{\ln M(G,t)}{|G|} - \frac{1}{2}p(G,t)\ln(t).$$

Note that

$$\tilde{\lambda}(G) = F(G,1)$$

is called the monomer-dimer free energy.

The function $p = p(G,t)$ is a strictly monotone increasing function which maps $[0,\infty)$ to $[0,p^*)$, where $p^* = \frac{2\nu(G)}{|G|}$, where $\nu(G)$ denotes the number of edges in the largest matching. If G contains a perfect matching, then $p^* = 1$. Therefore, its inverse function $t = t(G,p)$ maps $[0,p^*)$ to $[0,\infty)$. (If G is clear from the context, then we will simply write $t(p)$ instead of $t(G,p)$.) Let

$$\lambda_G(p) = F(G,t(p))$$

if $p < p^*$, and $\lambda_G(p) = 0$ if $p > p^*$. Note that we have not defined $\lambda_G(p^*)$ yet. We simply define it as a limit:

$$\lambda_G(p^*) = \lim_{p \nearrow p^*} \lambda_G(p).$$

We will show that this limit exists, see part (d) of Proposition 3.15. Later we will extend the definition of $p(G,t), F(G,t)$ and $\lambda_G(p)$ to infinite lattices L.

The intuitive meaning of $\lambda_G(p)$ is the following. Assume that we want to count the number of matchings covering p fraction of the vertices. Let us assume that it makes sense: $p = \frac{2k}{|G|}$, and so we wish to count $m_k(G)$. Then

$$\lambda_G(p) \approx \frac{\ln m_k(G)}{|G|}.$$

For the more precise formulation of this statement, see Proposition 3.15.

Our next aim is to extend the definition of the function $\lambda_G(p)$ for infinite lattices L. We also show an efficient way of computing its values if p is sufficiently separated from p^*.

Theorem 3.16. *Let (G_n) be a Benjamini–Schramm convergent sequence of bounded degree graphs. Then the sequences of functions*
(a)

$$p(G_n, t),$$

(b)

$$\frac{\ln M(G_n, t)}{|G_n|}$$

converge to strictly monotone increasing continuous functions on the interval $[0, \infty)$.
If, in addition, every G_n has a perfect matching then the sequences of functions
(c)

$$t(G_n, p),$$

(d)

$$\lambda_{G_n}(p)$$

are convergent for all $0 \le p < 1$.

Definition 3.18. *Let L be an infinite lattice and (G_n) be a sequence of finite graphs which is Benjamini–Schramm convergent to L. For instance, G_n can be chosen to be an exhaustion of L. Then the sequence of measures (ρ_{G_n}) weakly converges to some measure which we will call ρ_L, the matching measure of the lattice L. For $t > 0$, we can introduce*

$$p(L,t) = \int \frac{tz^2}{1+tz^2}\, d\rho_L(z)$$

and

$$F(L,t) = \int \frac{1}{2} \ln\left(1 + tz^2\right)\, d\rho_L(z) - \frac{1}{2} p(L,t) \ln(t).$$

The monomer-dimer free energy of a lattice L is

$$\tilde{\lambda}(L) = F(L,1) = \int \frac{1}{2} \ln\left(1+z^2\right)\, d\rho_L(z)$$

The following table contains some numerical results.

Lattice	$\tilde{\lambda}(L)$	Bound on error	$p(L,1)$	Bound on error
2d	0.6627989725	$3.72 \cdot 10^{-8}$	0.638123105	$5.34 \cdot 10^{-7}$
3d	0.7859659243	$9.89 \cdot 10^{-7}$	0.684380278	$1.14 \cdot 10^{-5}$
4d	0.8807178880	$5.92 \cdot 10^{-6}$	0.715846906	$5.86 \cdot 10^{-5}$
5d	0.9581235802	$4.02 \cdot 10^{-5}$	0.739160383	$3.29 \cdot 10^{-4}$
6d	1.0237319240	$1.24 \cdot 10^{-4}$	0.757362382	$8.91 \cdot 10^{-4}$
7d	1.0807591953	$3.04 \cdot 10^{-4}$	0.772099489	$1.95 \cdot 10^{-3}$
hex	0.58170036638	$1.56 \cdot 10^{-9}$	0.600508638	$2.65 \cdot 10^{-8}$

In general, the matching measure ρ_L can contain atoms. For instance, if G is a finite graph then clearly ρ_G consists of atoms. On the other hand, it can be shown that for all lattices in the table above, the measure ρ_L is atomless.

Theorem 3.22. *Let L be a lattice satisfying one of the following conditions.*

(a) The lattice L can be obtained as a Benjamini–Schramm limit of a finite graph sequence G_n such that G_n can be covered by $o(|G_n|)$ disjoint paths.

(b) The lattice L can be obtained as a Benjamini–Schramm limit of connected vertex transitive finite graphs.

Then the matching measure ρ_L is atomless.

Results from Chapter 4:

We call the graph G *positive* if $\hom(G,H) \geq 0$ for every edge-weighted graph H (where the edgeweights may be negative). It would be interesting to characterize these graphs; in this chapter we offer a conjecture and line up supporting evidence.

We call a graph *symmetric*, if its vertices can be partitioned into three sets (S,A,B) so that S is an independent set, there is no edge between A and B, and there exists an isomorphism between the subgraphs spanned by $S \cup A$ and $S \cup B$ which fixes S.

Conjecture 4.1. *A graph G is positive if and only if it is symmetric.*

The "if" part of the conjecture is easy.

Lemma 4.2. *If a graph G is symmetric, then it is positive.*

In the reverse direction, we only have partial results. We are going to prove that the conjecture is true for trees (Corollary 4.20) and for all graphs up to 9 nodes (see Section 4.5).

We state and prove a number of properties of positive graphs. Each of these is of course satisfied by symmetric graphs.

Lemma 4.3. *If G is positive, then G has an even number of edges.*

We call a homomorphism *even* if the preimage of each edge is has even cardinality.

Lemma 4.4. *If G is positive, then there exists an even homomorphism of G into itself.*

For two looped-simple graphs G_1 and G_2, we denote by $G_1 \times G_2$ their *categorical product*, defined by

$$\begin{aligned} &V(G_1 \times G_2) = V(G_1) \times V(G_2), \\ &E(G_1 \times G_2) = \big\{\big((i_1, i_2), (j_1, j_2)\big) : \ (i_1, j_1) \in E(G_1), (i_2, j_2) \in E(G_2)\big\}. \end{aligned}$$

Let K_n^+ denote the complete graph on the vertex set $[n]$ with loops at all vertices, where $n \geq |V(G)|$.

Theorem 4.5. *If a graph G is positive, then there exists an even homomorphism $f : G \to K_n^+ \times G$ so that $\big|f(V(G))\big| \geq \frac{1}{2}\big|V(G)\big|$.*

We develop a technique to show that one can partition the vertices of a positive graph in a certain way so that subgraphs spanned by each part are also positive. The main idea is to limit, over what maps $p : V \to [0, 1]$ one has to average to check positivity. Using this idea recursively we can finally reduce to maps that take each partition to disjoint subsets of $[0, 1]$. This in turn allows us to conclude positivity of the spanned subgraphs.

To this end, first we have to introduce the notion of $\mathcal{F}$-positivity. Let $G = (V, E)$ be a simple graph. For a measurable subset $\mathcal{F} \subseteq [0, 1]^V$ and a bounded measurable weight function $\omega : \ [0, 1] \to (0, \infty)$, we define

$$t(G, W, \omega, \mathcal{F}) = \int\limits_{p \in \mathcal{F}} \hom(G, W, \omega, p) \, \mathrm{d}p, \tag{1}$$

where the *weight* of a $p : V \to [0, 1]$ is

$$\hom(G, W, \omega, p) = \prod_{v \in V} \omega\big(p(v)\big) \prod_{e \in E} W\big(p(e)\big) \tag{2}$$

With the measure μ with density function ω (i.e., $\mu(X) = \int_X \omega$), we can write this as

$$t(G, W, \omega, \mathcal{F}) = \int\limits_{\mathcal{F}} \prod_{e \in E} W\big(p(e)\big) \, \mathrm{d}\mu^V(p). \tag{3}$$

We say that G is *$\mathcal{F}$-positive* if for every kernel W and function ω as above, we have $t(G, W, \omega, \mathcal{F}) \geq 0$. It is easy to see that G is $[0, 1]^V$-positive if and only if it is positive.

For a partition $\mathcal{P}$ of $[0, 1]$ into a finite number of sets with positive measure and a function $\pi : \ V \to \mathcal{P}$, we call the box $\mathcal{F}(\pi) = \{p \in [0, 1]^V : \ p(v) \in \pi(v) \ \forall v \in V\}$ a *partition-box*. Equivalently, a partition-box is a product set $\prod_{v \in V} S_v$, where the sets $S_v \subseteq [0, 1]$ are measurable, and either $S_u \cap S_v = \emptyset$ or $S_u = S_v$ for all $u, v \in V$.

A partition $\mathcal{N}$ of V is *positive* if for any partition $\mathcal{P}$ as above, and any $\pi : V \to \mathcal{P}$ such that $\pi^{-1}(\mathcal{P}) = \mathcal{N}$, G is $\mathcal{F}(\pi)$-positive.

The *walk-tree* of a rooted graph (G, v) is the following infinite rooted tree $R(G, v)$: its nodes are all finite walks starting from v, its root is the 0-length walk, and the parent of any other walk is obtained by deleting its last node. The *walk-tree partition* $\mathcal{R}$ is the partition of V in which two nodes $u, v \in V$ belong to the same class if and only if $R(G, u) \cong R(G, v)$.

Proposition 4.16. *If a graph G is positive, then its walk-tree partition is also positive.*

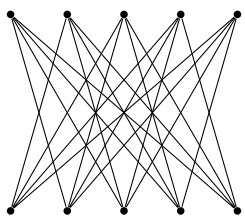
Corollary 4.17. *Let $G(V, E)$ be a positive graph, and let $S \subset V$ be the union of some classes of the walk-tree partition. Then $G[S]$ is also positive.*

Corollary 4.18. *If G is positive, then for each k the subgraph of G spanned by all nodes with degree k is positive as well.*

Corollary 4.19. *For each odd k the number of nodes of G with degree k must be even.*

Corollary 4.20. *Conjecture 4.1 is true for trees.*

We checked Conjecture 4.1 for all graphs on at most 10 vertices using the previous results and a computer program. We verified that all positive graphs are symmetric, possibly except the one below, which is not symmetric, but we could not decide whether it is positive.



Bibliography

[1] M. Abért, P. Csikvári, P. E. Frenkel and G. Kun: *Matchings in Benjamini–Schramm convergent graph sequences*, arXiv:1405.3271, to appear in Trans. Amer. Math. Soc.

[2] M. Abért and T. Hubai: *Benjamini–Schramm convergence and the distribution of chromatic roots for sparse graphs*, arXiv:1201.3861, to appear in Combinatorica

[3] M. Abért, P. Csikvári and T. Hubai: *Matching measure, Benjamini–Schramm convergence and the monomer-dimer free energy*, arXiv:1405.6740

[4] M. Abért, A. Thom and B. Virág: *Benjamini-Schramm convergence and pointwise convergence of the spectral measure*, preprint at `http://www.renyi.hu/~abert/`

[5] S. E. Alm: *Upper bounds for the connective constant of self-avoiding walks*, Comb. Probab. Comp. **2**, pp. 115–136

[6] A. Bandyopadhyay and D. Gamarnik, *Counting without sampling. Asymptotics of the logpartition function for certain statistical physics models*, Random Structures & Algorithms 33 (2008)

[7] R. J. Baxter: *Dimers on a rectangular lattice*, J. Math Phys. **9** (1968), pp. 650–654

[8] I. Benjamini and O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. 6 (2001), no. 23, 13 pp.

[9] G. D. Birkhoff: *A Determinant Formula for the Number of Ways of Coloring a Map*, Annals of Mathematics 14 (1912) 42–46.

[10] C. Borgs, *Absence of Zeros for the Chromatic Polynomial on Bounded Degree Graphs*, Combinatorics, Probability and Computing 15 (2006), 63–74.

[11] C. Borgs, J. Chayes, J. Kahn and L. Lovász, *Left and right convergence of graphs with bounded degree*, Random Structures & Algorithms 42 (2013)

[12] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: *Convergent Graph Sequences I: Subgraph frequencies, metric properties, and testing*, Advances in Math. **219** (2008), 1801–1851.

[13] P. Butera, P. Federbush and M. Pernici: *Higher order expansion for the entropy of a dimer or a monomer-dimer system on d-dimensional lattices*, Physical Review E**87**, 062113 (2013)

[14] P. Butera and M. Pernici: *Yang-Lee edge singularities from extended activity expansions of the dimer density for bipartite lattices of dimensionality* $2 \leq d \leq 7$, Physical Review E**86**, 011104 (2012)

[15] O. A. Camarena, E. Csóka, T. Hubai, G. Lippner and L. Lovász: *Positive graphs*, arXiv:1205.6510, to appear in European J. Comb.

[16] P. Csikvári and P. E. Frenkel, *Benjamini–Schramm continuity of root moments of graph polynomials*, arXiv:1204.0463

[17] J. N. Darroch: *On the distribution of the number of successes in independent trials*, Ann. Math. Statist. **35**, pp. 1317–1321

[18] B. L. Douglas: *The Weisfeiler-Lehman Method and Graph Isomorphism Testing*, arXiv:1101.5211

[19] H. Duminil-Copin and S. Smirnov: *The connective constant of the honeycomb lattice equals* $\sqrt{2+\sqrt{2}}$, Ann. of Math. (2) **175**(3), pp. 1653–1665

[20] G. Elek and G. Lippner: *Borel oracles. An analytical approach to constant-time algorithms*, Proc. Amer. Math. Soc. **138**(8) (2010), pp. 2939–2947.

[21] M. E. Fisher: *Statistical mechanics of dimers on a plane lattice*, Phys. Rev. **124** (1961), pp. 1664–1672

[22] S. Friedland and L. Gurvits, *Lower bounds for partial matchings in regular bipartite graphs and applications to the monomer-dimer entropy*, Combinatorics, Probability and Computing 17 (2008), 347-361.

[23] S. Friedland and U. N. Peled: *Theory of Computation of Multidimensional Entropy with an Application to the Monomer-Dimer Problem*, Advances of Applied Math. **34** (2005), pp. 486–522

[24] D. Gamarnik and D. Katz: *Sequential cavity method for computing free energy and surface pressure*, J Stat Phys **137** (2009), pp. 205–232

[25] C. D. Godsil: *Algebraic Combinatorics*, Chapman and Hall, New York 1993

[26] L. Gurvits: *Unleashing the power of Schrijver's permanental inequality with the help of the Bethe Approximation*, arXiv:1106.2844v11

[27] T. Hara, G. Slade and A. D. Sokal: *New lower bounds on the self-avoiding-walk connective constant*, J. Statist. Phys. **72** (1993), pp. 479–517

[28] J. M. Hammersley: *Existence theorems and Monte Carlo methods for the monomer-dimer problem.* In: David, F.N. (ed.) Research Papers in Statistics: Festschrift for J. Neyman, pp. 125–146. Wiley, London (1966)

[29] J. M. Hammersley: *An improved lower bound for the multidimensional dimer problem.* Proc. Camb. Philos. Soc. **64** (1966), pp. 455–463

[30] J. M. Hammersley, V. Menon: *A lower bound for the monomer-dimer problem.* J. Inst. Math. Appl. **6** (1970), pp. 341–364

[31] H. Hatami: *Graph norms and Sidorenko's conjecture*, Israel J. Math. **175** (2010), 125–150.

[32] O. J. Heilmann and E. H. Lieb: *Theory of monomer-dimer systems*, Commun. Math. Physics **25** (1972), pp. 190–232

[33] Y. Huo, H. Liang, S. Q. Liu, F. Bai: *Computing the monomer-dimer systems through matrix permanent*, Phys. Rev. E **77** (2008)

[34] P. W. Kasteleyn: *The statistics of dimers on a lattice, I: the number of dimer arrangements on a quadratic lattice*, Physica **27** (1961), pp. 1209–1225

[35] C. Y. Ku and W. Chen: *An analogue of the Gallai–Edmonds Structure Theorem for non-zero roots of the matching polynomial*, Journal of Combinatorial Theory, Series B **100** (2010), pp. 119–127

[36] L. Lovász: *Large Networks and Graph Limits*, Colloquium Publications, vol. 60. American Mathematical Society (2012)

[37] L. Lovász: *Subgraph densities in signed graphons and the local Sidorenko conjecture*, arXiv:1004.3026

[38] L. Lovász, B. Szegedy: *Limits of dense graph sequences*, J. Comb. Theory B **96** (2006), 933–957.

[39] R. Lyons, *Asymptotic enumeration of spanning trees*, Combinatorics, Probability and Computing 14 (2005), 491–522.

[40] S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk (N.S.) 7 (48) (1952), 31–122.

[41] H. N. Nguyen and K. Onak, *Constant-time approximation algorithms via local improvements*, 49th Annual IEEE Symposium on Foundations of Computer Science (2008), pp. 327–336.

[42] A. Procacci, B. Scoppola and V. Gerasimov: *Potts model on infinite graphs and the limit of chromatic polynomials.* Commun. Math. Phys. 235 (2003), 215–231.

[43] G. C. Rota, *On the foundations of combinatorial theory I. Theory of Möbius functions*, Probability theory and related fields 2 (4) (1964), 340–368.

[44] H. N. V. Temperley and M. E. Fisher: *Dimer problem in statistical mechanics–an exact result*, Philos. Mag. **6** (1961), pp. 1061–1063

[45] J. Salas and A.D. Sokal, *Transfer Matrices and Partition-Function Zeros for Antiferromagnetic Potts Models V. Further Results for the Square-Lattice Chromatic Polynomial*, J Stat Phys 135 (2009), 279–373.

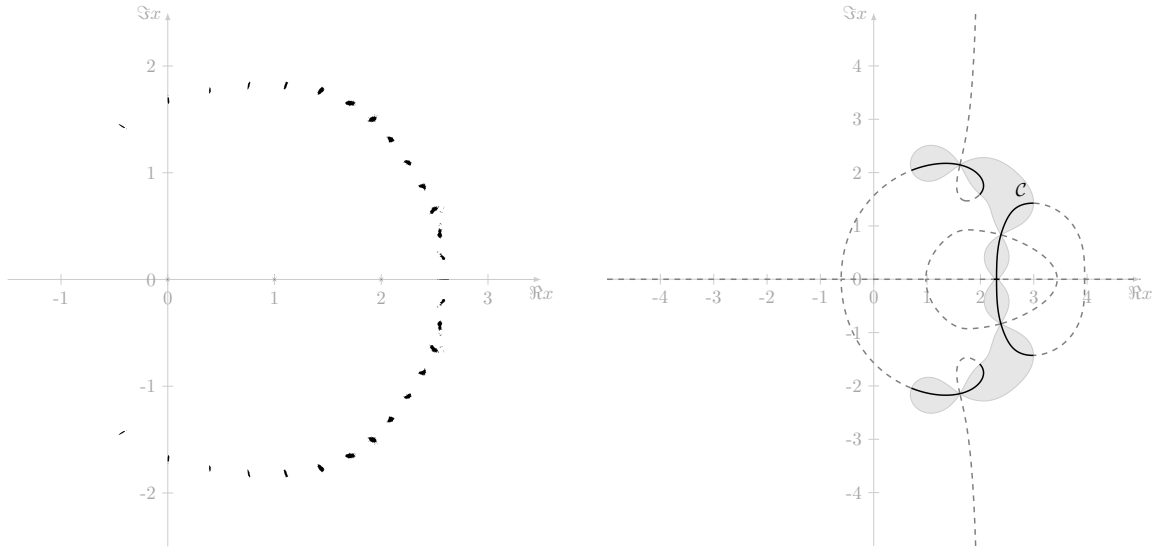
[46] A. D. Sokal: *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions*, Combinatorics, Probability and Computing 10 (2001), 41–77.

[47] A. D. Sokal, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, In: Webb, BS, (ed.) Surveys in Combinatorics, 2005, 173–226. Cambridge University Press

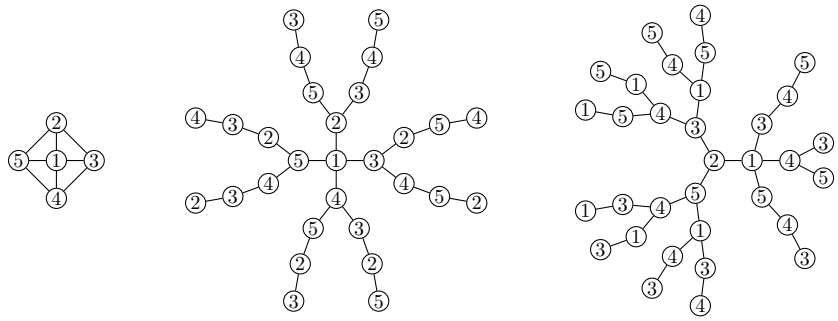
[48] A. Thom: *Sofic groups and diophantine approximation*, Comm. Pure Appl. Math., Vol. LXI, (2008), 1155–1171

[49] B. Weisfeiler and A.A. Lehman: *A reduction of a graph to a canonical form and an algebra arising during this reduction (in Russian)*, Nauchno-Technicheskaya Informatsia, Seriya 2, **9** (1968), 12–16.

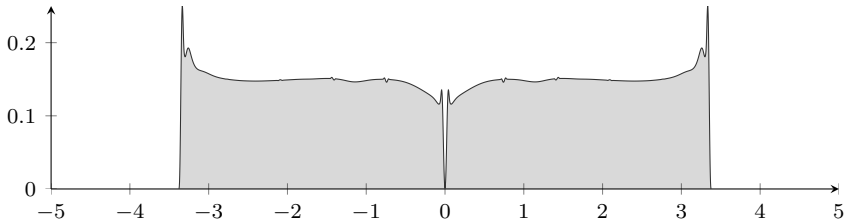
Figures



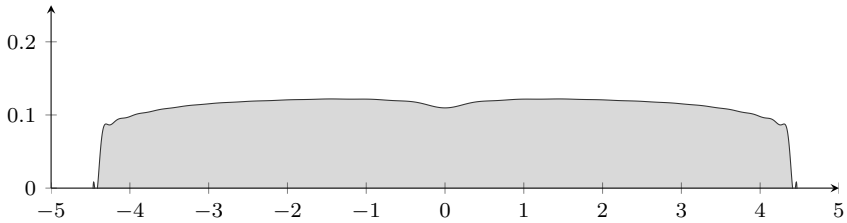
Left: chromatic roots of the 30368 cubic graphs of size 32 and girth 7
Right: possible limit points of chromatic roots of $T_n = C_4 \times P_n$ as $n \to \infty$



The pyramid graph and its trees of self-avoiding walks starting from ① and ②



An approximation for the matching measure of $\mathbb{Z}^2$



An approximation for the matching measure of $\mathbb{Z}^3$