Maximum Principles in the Theory of Numerical Methods

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Theses of the Ph.D. Dissertation

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Part I

Basic notions of numerical analysis

1 Introduction

The topic of the first part is the Lax theory of the numerical solution of linear and nonlinear equations. To approximate the solution of some equation, usually a numerical method is used, the success of which depends on its convergence. The definition of convergence is theoretical since it contains the unknown solution, however, this problem can be solved with the following idea. The directly unverifiable notion of convergence can be substituted with the notions of consistency and stability. In the linear case stability and convergence are equivalent under the consistency assumption, this is the Lax equivalence theorem.

In the first part of the dissertation, our goal was to present a framework that unifies the known results, completes the theory and clarifies the relations between the basic notions of consistency, stability and convergence.

Used methods. We used Stetter's lemma [7, Lem. 1.2.1.] (which result is based on the Brouwer's invariance domain theorem); density argumentation; theory of FEM and FDM; and we built upon the Lax equivalence theorem presented in the form given in [6, Palencia and Sanz-Serna, 1985] (which result is based on the Banach-Steinhaus theorem).

2 Results

1. We built up an abstract framework on the approximation of nonlinear equations. We presented a Lax-type theorem which is suitable for applications. The listed results are based on the paper [1, Faragó, Mincsovics, Fekete, 2012]. For the details we recommend this paper or the dissertation.

   Thm. 1.1.36. We proved that if the numerical process is densely consistent and stable then it is convergent as well (under assumptions (a1)–(a3) of Ass. 1.1.9. and (a4)–(a6) of Ass. 1.1.33.), and the order of the convergence can be estimated from below by the order of consistency on some corresponding set.

   Lem. 1.1.37. We stated that it is sufficient to check stability on a set of elements that the union of their stability neighbourhoods contains the restriction-image of the solution and the infimum of their stability constants is positive.

   Checking consistency on a set of elements can be done in parallel, thus Thm. 1.1.36. and Lem. 1.1.37. together provide the opportunity for using our nonlinear framework in applications.

   Ex. 1.1.16., 30., 31., 32., 39., 40., 41. We showed the relations of the basic notions with the help of numerous examples.

2. In **Subsection 1.1.2.** we showed what stability means when FEM is applied on an linear elliptic problem, and when FEM + θ-method is applied on a linear parabolic problem.

Part II

Discrete maximum principles

3 Introduction

The second part of the dissertation deals with discrete elliptic and parabolic maximum principles. When choosing a numerical method to approximate the solution of a continuous mathematical problem, the first thing to consider is which method results in a good approximation from a quantitative point of view. This was investigated in the first part of the thesis. However, in most of the cases it is not enough. The original problem (which is usually some model of a phenomenon) possesses important qualitative properties, and a natural requirement from the numerical solution is to preserve these qualitative properties. E.g., when we seek an approximation of the Laplace's equation where the boundary condition is defined to be nonnegative then the solution is nonnegative, too and a good approximation should be nonnegative as well. For linear elliptic and parabolic problems the main qualitative properties are the various maximum principles.

In Chapter 3, which dealt with discrete elliptic maximum principles, our aim was twofold. Firstly, we wanted to present a unified algebraic framework giving the known results and completing the theory with our results on discrete strong maximum principles. Secondly, we wanted to apply this framework on a certain problem.

In Chapter 4, which dealt with discrete parabolic maximum principles, our aim was the following. Firstly, to present an algebraic framework on discrete parabolic maximum principles collecting the known results. Next, we wanted to apply this framework on a certain practical problem. Finally, we also wanted to find some connection between discrete elliptic and discrete parabolic maximum principles.

Used methods. Theory of FEM and FDM; geometry of simplices; linear algebra; Z- and M-matrix theory; inverse-nonnegative matrices; nonnegative matrices; Perron-Frobenius theory; matrix-splitting theory.

4 Preliminaries – definitions, notations, problem settings

Discrete elliptic maximum principles. We defined discrete elliptic maximum principles for a matrix $\mathbf{K} = (\mathbf{K}_0|\mathbf{K}_\partial) \in \mathbb{R}^{N\times\overline{N}}$ acting on the vector $\mathbf{u} = (\mathbf{u}_0|\mathbf{u}_\partial)^T \in \mathbb{R}^{\overline{N}}$, where $\mathbf{u}_0 \in \mathbb{R}^N$, $\mathbf{u}_\partial \in \mathbb{R}^{N_\partial}$.

Def. 3.1.1. and 2. We say that the matrix $\mathbf{K}$ possesses

- the *discrete weak non-positivity preservation property* (DnP) if the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0}\,, \quad \max \mathbf{u}_\partial \leq 0 \quad \Rightarrow \quad \max \mathbf{u} \leq 0\,.$$

- the *discrete strong non-positivity preservation property* (DNP) if it possesses the DnP, moreover, the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0} \quad \text{and} \quad \max \mathbf{u} = \max \mathbf{u}_0 = 0 \quad \Rightarrow \quad \mathbf{u} = \mathbf{0}\,.$$

- the *discrete weak maximum principle* (DwMP) if the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0} \quad \Rightarrow \quad \max \mathbf{u} \leq \max\{0, \mathbf{u}_\partial\}\,;$$

- the *discrete strictly weak maximum principle* (DWMP) if the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0} \quad \Rightarrow \quad \max \mathbf{u} = \max \mathbf{u}_\partial\,;$$

- the *discrete strong maximum principle* (DsMP) if it possesses the DwMP, moreover, the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0} \quad \text{and} \quad \max \mathbf{u} = \max \mathbf{u}_0 = m \geq 0 \quad \Rightarrow \quad \mathbf{u} = m\mathbf{e}\,;$$

- the *discrete strictly strong maximum principle* (DSMP) if it possesses the DWMP, moreover, the following implication holds:
$$\mathbf{Ku} \leq \mathbf{0} \quad \text{and} \quad \max \mathbf{u} = \max \mathbf{u}_0 = m \quad \Rightarrow \quad \mathbf{u} = m\mathbf{e}\,.$$

Problem 1. Consider the elliptic operator K, where
$$Ku = -(pu')' + k^2 u\,, \tag{1}$$
where $\Omega = (0,1)$, $\operatorname{dom} K = H^1(0,1)$, $p, k \in \mathbb{R}$, $p > 0$.

We apply the interior penalty discontinuous Galerkin method. The first step is to define a mesh on $(0,1)$. Let us denote it by τ_h and define it in the following way: $0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = 1$. We use the notations $I_n = [x_{n-1}, x_n]$, $h_n = |I_n|$, $h_{n-1,n} = \max\{h_{n-1}, h_n\}$, (with $h_{0,1} = h_1$, $h_{N,N+1} = h_N$).

The next step is to define the space $D_l(\tau_h) = \{v : v|_{I_n} \in P_l(I_n), \forall n = 1, 2, \ldots, N\}$ – piecewise polynomials over every interval with maximal degree l. For these functions we introduce the right and left hand side limits $v(x_n^+) = \lim\limits_{t\to 0^+} v(x_n + t)$, $v(x_n^-) = \lim\limits_{t\to 0^+} v(x_n - t)$, and jumps and averages over the mesh nodes as
$$[\![u(x_n)]\!] = u(x_n^-) - u(x_n^+)\,, \quad \{\!\!\{u(x_n)\}\!\!\} = \frac{1}{2}(u(x_n^-) + u(x_n^+))\,.$$
At the boundary nodes these are defined as
$$[\![u(x_0)]\!] = -u(x_0^+)\,,\ \{\!\!\{u(x_0)\}\!\!\} = u(x_0^+)\,,\ [\![u(x_N)]\!] = u(x_N^-)\,,\ \{\!\!\{u(x_N)\}\!\!\} = u(x_N^-)\,.$$

We fix the penalty parameter $\sigma \geq 0$ and ε, which can be any arbitrary number, but it is usually chosen from the set $\{-1, 0, 1\}$.

After these preparations we are ready to define the (discrete) IPDG bilinear form as

$$a_{DG}(u,v) = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} pu'(x)v'(x)\,\mathrm{d}x - \sum_{n=0}^{N} \{\!\{pu'(x_n)\}\!\} [\![v(x_n)]\!] + \\ \varepsilon \sum_{n=0}^{N} \{\!\{pv'(x_n)\}\!\} [\![u(x_n)]\!] + \sum_{n=0}^{N} \frac{\sigma}{h_{n,n+1}} [\![v(x_n)]\!] [\![u(x_n)]\!] + \int_0^1 k^2 uv\,\mathrm{d}x\,. \tag{2}$$

The next step is the following. We fix a basis in the space $D_l(\tau_h)$. We chose $l = 1$. We will use $\Phi_i^1(x)$ for the $(2(i-1)+1)$th basis functions, and $\Phi_i^2(x)$ for the $(2(i-1)+2)$th basis function. On interval I_i the function $\Phi_i^1(x)$ is the linear function with $\Phi_i^1(x_{i-1}^+) = 1$, $\Phi_i^1(x_i^-) = 0$ and $\Phi_i^2(x)$ is the linear function with $\Phi_i^1(x_{i-1}^+) = 0$, $\Phi_i^1(x_i^-) = 1$, and these functions are zero outside I_i.

Finally, we construct the IPDG elliptic operator $\mathbf{K} = (\mathbf{K}_0|\mathbf{K}_\partial)$ using the bilinear form (2), similarly to the standard FEM approach. However, there are small differences since here $\mathbf{K} \in \mathbb{R}^{(2N-2)\times(2N)}$, $\mathbf{K}_0 \in \mathbb{R}^{(2N-2)\times(2N-2)}$, and $\mathbf{K}_\partial \in \mathbb{R}^{(2N-2)\times 2}$. The $2N$ basis function are ordered as follows: the first $2N-2$ are the basis functions that belong to the interior nodes and they are numbered from left to right. The $(2N-1)$th belongs to the left boundary and the $2N$th belongs to the right boundary.

Discrete parabolic maximum principles and discrete stabilization property. We defined discrete maximum principles for a hyper-matrix $\mathcal{L}$ defined as $(\mathcal{L}\nu)^0 = \mathbf{v}^0$, $(\mathcal{L}\nu)^n = \mathbf{X}_1\mathbf{v}^n - \mathbf{X}_2\mathbf{v}^{n-1}$, $n = 1, \dots, M$, where $\mathbf{X}_1 = (\mathbf{X}_{10}|\mathbf{X}_{1\partial})$, $\mathbf{X}_2 = (\mathbf{X}_{20}|\mathbf{X}_{2\partial}) \in \mathbb{R}^{N\times\overline{N}}$; $\mathbf{X}_{10}, \mathbf{X}_{20} \in \mathbb{R}^{N\times N}$; $\mathbf{X}_{1\partial}, \mathbf{X}_{2\partial} \in \mathbb{R}^{N\times N_\partial}$, $\overline{N} = N + N_\partial$. $(\nu)^n = \mathbf{v}^n = (\mathbf{v}_0^n|\mathbf{v}_\partial^n)^T \in \mathbb{R}^{\overline{N}}$, $\mathbf{v}_0^n \in \mathbb{R}^N$, $\mathbf{v}_\partial^n \in \mathbb{R}^{N_\partial}$ in **Def.4.1.1.**. It is known that the discrete parabolic maximum principles can be characterized, we appended this result in the following lemma.

Lem. 4.1.3. The hyper-matrix $\mathcal{L}$ possesses

- the discrete non-positivity preservation property (DnP) if and only if (for all $\mathbf{v}^n, \mathbf{v}^{n-1}$) the following implication holds.

$$(\mathcal{L}\nu)^n \equiv \mathbf{X}_1\mathbf{v}^n - \mathbf{X}_2\mathbf{v}^{n-1} \leq \mathbf{0}\,, \quad \max\{\mathbf{v}^{n-1}, \mathbf{v}_\partial^n\} \leq 0 \quad \Rightarrow \quad \max \mathbf{v}^n \leq 0\,;$$

- the discrete maximum principle (DmP) if and only if (for all $\mathbf{v}^n, \mathbf{v}^{n-1}$) the following implication holds.

$$(\mathcal{L}\nu)^n \equiv \mathbf{X}_1\mathbf{v}^n - \mathbf{X}_2\mathbf{v}^{n-1} \leq \mathbf{0} \quad \Rightarrow \quad \max \mathbf{v}^n \leq \max\{0, \mathbf{v}^{n-1}, \mathbf{v}_\partial^n\}\,;$$

- the discrete strict maximum principle (DMP) if and only if (for all $\mathbf{v}^n, \mathbf{v}^{n-1}$) the following implication holds.

$$(\mathcal{L}\nu)^n \equiv \mathbf{X}_1\mathbf{v}^n - \mathbf{X}_2\mathbf{v}^{n-1} \leq \mathbf{0} \quad \Rightarrow \quad \max \mathbf{v}^n \leq \max\{\mathbf{v}^{n-1}, \mathbf{v}_\partial^n\}\,.$$

We employed the discrete stabilization property in the dissertation in order to find some connection between the discrete elliptic and parabolic maximum principles. Let $\mathbf{K}$ be defined as $\mathbf{K} = \mathbf{X}_1 - \mathbf{X}_2$. If $\mathbf{X}_{10}$ is non-singular, then we can write

$$\mathbf{v}_0^n = \mathbf{X}_{10}^{-1}\mathbf{X}_{20}\mathbf{v}_0^{n-1} + \mathbf{X}_{10}^{-1}\mathbf{X}_2\mathbf{v}_\partial^{n-1} - \mathbf{X}_{10}^{-1}\mathbf{X}_{1\partial}\mathbf{v}_\partial^n + \mathbf{X}_{10}^{-1}(\mathcal{L}\nu)^n\,, \quad n = 1, \dots .$$

Def. 4.3.1. The hyper-matrix $\mathcal{L}$ possesses the *discrete stabilization property* (DSP) if $\mathbf{K}_0$ is non-singular and for all $\mathbf{u}$, $\mathbf{v}_0^0$ the iteration

$$\mathbf{X}_1\mathbf{v}^n - \mathbf{X}_2\mathbf{v}^{n-1} = \mathbf{K}\mathbf{u}\,, \quad \mathbf{v}_\partial^{n-1} = \mathbf{u}_\partial\,, \quad n = 1, \dots$$

is convergent, moreover

$$\mathbf{v}^n \to \mathbf{u}$$

holds.

Problem 2. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain that can be covered by a regular simplicial mesh $\mathcal{T}_h$ with the property that this mesh is of nonobtuse type, i.e., all the angles made by any faces of each simplex $S \in \mathcal{T}_h$ are not greater than $\pi/2$.

We consider the parabolic operator which is defined for the functions $v(x,t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and which can be described as

$$L_{a,b,c}v = \frac{\partial v}{\partial t} - \mathrm{div}(a \text{ grad } v) + \langle b, \text{grad } v\rangle + c\, v, \tag{3}$$

where $a, c : \Omega \to \mathbb{R}$, $b : \Omega \to \mathbb{R}^d$, $a, b, c \in C(\overline{\Omega})$ and $a \in C^1(\Omega)$. The symbol $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^d$. In the sequel we assume that $0 < a_m \le a \le a_M$, $\|b\| \le b_M$ and $0 \le c \le c_M$ holds with the constants a_m, a_M, b_M, c_M. $\|\cdot\|$ denotes the norm of $\mathbb{R}^d$ induced by the scalar product $\langle \cdot, \cdot \rangle$.

Using the FEM+θ-method, where we cover Ω by a regular simplicial mesh $\mathcal{T}_h$ and we use the usual hat functions we arrive at the form $\mathbf{X}_1\mathbf{v}^{n+1} - \mathbf{X}_2\mathbf{v}^n$ which will be denoted by $\mathcal{L}_{a,b,c}$ or $\mathcal{L}_{a,b,0}$ if our starting point was the operator $L_{a,b,c}$ or $L_{a,b,0}$, with the roles $\mathbf{X}_1 = \frac{1}{\Delta t}\mathbf{M} + \theta\mathbf{K}$, $\mathbf{X}_2 = \frac{1}{\Delta t}\mathbf{M} - (1-\theta)\mathbf{K}$, where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices determined by the bilinear forms

$$(\mathbf{M})_{ij} = B_1(\phi_j, \phi_i) = \int_\Omega \phi_j\phi_i \, \mathrm{d}\mathbf{x}\,,$$

$$(\mathbf{K})_{ij} = B_2(\phi_j, \phi_i) = \int_\Omega a\ \langle \mathrm{grad}\phi_j, \mathrm{grad}\phi_i\rangle\ \mathrm{d}\mathbf{x} + \int_\Omega \langle b, \mathrm{grad}\phi_j\rangle\, \phi_i \, \mathrm{d}\mathbf{x} + \int_\Omega c\phi_j\phi_i \, \mathrm{d}\mathbf{x}\,,$$

where $i = 1, \dots, N\,,\ j = 1, \dots, \overline{N}$, respectively.

If the simplex S is tightened by the $d+1$ piece vertices $\mathbf{x}_i$, and we denote by S_i the $(d-1)$-dimensional face opposite to the vertex $\mathbf{x}_i$, then $\cos\gamma_{ij}$ is the cosine of the interior angle between faces S_i and S_j. Note that $(\mathrm{meas_d S})d = (\mathrm{meas_{d-1} S_i})m_i$, where m_i is the (Euclidean) distance between S_i and $\mathbf{x}_i$.

5 Results

1. We presented an algebraic framework on discrete elliptic maximum principles in which we characterized the discrete strong maximum principles. These results are based on the paper [5, Mincsovics and Horváth, 2012].

   Lem. 3.1.5. We assume that $N \geq 2$. The matrix $\mathbf{K}$ possesses the DNP if and only if the following two conditions hold:

   (N1) $\mathbf{K}_0^{-1} > \mathbf{0}\,;$ (N2) $-\mathbf{K}_0^{-1}\mathbf{K}_\partial > \mathbf{0}\,.$

   Thm. 3.1.9. We assume that $N \geq 2$. The matrix $\mathbf{K}$ possesses the DSMP if and only if the following three conditions hold:

   (S1) $\mathbf{K}_0^{-1} > \mathbf{0}\,;$ (S2) $-\mathbf{K}_0^{-1}\mathbf{K}_\partial > \mathbf{0}\,;$ (S3) $-\mathbf{K}_0^{-1}\mathbf{K}_\partial\mathbf{e} = \mathbf{e}\,.$

   Thm. 3.1.10. We assume that $N \geq 2$. The matrix $\mathbf{K}$ possesses the DsMP if and only if the following three conditions hold:

   (s1) $\mathbf{K}_0^{-1} > \mathbf{0}\,;$ (s2) $-\mathbf{K}_0^{-1}\mathbf{K}_\partial > \mathbf{0}\,;$
   (s3) $-\mathbf{K}_0^{-1}\mathbf{K}_\partial\mathbf{e} < \mathbf{e}$ or $-\mathbf{K}_0^{-1}\mathbf{K}_\partial\mathbf{e} = \mathbf{e}\,.$

   In **Subsection 3.1.2.** we completed the list of practical algebraic conditions ensuring the discrete weak maximum principle and the discrete strong maximum principles, respectively.

   In **Subsection 3.1.3.** we showed the differences between the discrete weak and strong maximum principles with the help of numerical examples.

2. We considered some problem where IPDG discretization is used, see the details in the paragraph "Problem 1." . We investigated the preservation of discrete elliptic maximum principles. The following results are based on the paper [2, Horváth and Mincsovics, 2013].

   Thm. 3.3.2. Let $\mathbf{K} = (\mathbf{K}_0|\mathbf{K}_\partial)$ be the matrix constructed from (1) by the IPDG method as described earlier in paragraph "Problem 1.". This matrix has the DnP if we choose

   - ε as
   $$-1 \leq \varepsilon \leq 0\,, \qquad \text{when} \quad k = 0\,,$$
   $$-1 < \varepsilon \leq 0\,, \qquad \text{when} \quad k > 0\,,$$
   - σ as
   $$\frac{p(1-\varepsilon)}{2} \leq \sigma\,,$$

- the mesh $\mathcal{T}_h$ as

$$h_i^2 \le \frac{3p(\varepsilon+1)}{k^2}, \qquad i = 2,\dots,N-1, \qquad \text{(fineness at the interior)}$$

$$\begin{aligned} \frac{h_{i,i+1}}{h_{i+1}} - \frac{\varepsilon h_{i,i+1}}{h_i} &\le \frac{2\sigma}{p} \quad \text{and} \\ \frac{h_{i,i+1}}{h_i} - \frac{\varepsilon h_{i,i+1}}{h_{i+1}} &\le \frac{2\sigma}{p}, \quad i = 1,\dots,N-1. \end{aligned} \qquad \text{(uniformity)}$$

Thm. 3.3.3. Let $\mathbf{K} = (\mathbf{K}_0|\mathbf{K}_\partial)$ be the matrix constructed from (1) by the IPDG method as described earlier. This matrix possesses the DwMP if we choose

- ε as

$$-\frac{1}{2} \le \varepsilon \le 0, \qquad \text{when } k = 0,$$
$$-\frac{1}{2} < \varepsilon \le 0, \qquad \text{when } k > 0,$$

- σ as

$$\frac{p(1-\varepsilon)}{2} \le \sigma,$$

- the mesh $\mathcal{T}_h$ as

$$h_i^2 \le \frac{3p(2\varepsilon+1)}{k^2}, \qquad i = 1, N, \qquad \text{(fineness at the boundary)}$$

$$h_i^2 \le \frac{3p(\varepsilon+1)}{k^2}, \qquad i = 2,\dots,N-1, \qquad \text{(fineness at the interior)}$$

$$\begin{aligned} \frac{h_{i,i+1}}{h_{i+1}} - \frac{\varepsilon h_{i,i+1}}{h_i} &\le \frac{2\sigma}{p} \quad \text{and} \\ \frac{h_{i,i+1}}{h_i} - \frac{\varepsilon h_{i,i+1}}{h_{i+1}} &\le \frac{2\sigma}{p}, \quad i = 1,\dots,N-1. \end{aligned} \qquad \text{(uniformity)}$$

Rem. 3.3.4. We investigated the popular cases: $\varepsilon \in \{-1, 0, 1\}$.

Ex. 3.3.6. and 7. We investigated the sharpness of the conditions of Thms. 3.3.2 and 3.3.3. with numerical examples.

3. We investigated a parabolic problem when some FEM + θ-method discretization is used and we derived practical conditions under which the most important discrete parabolic maximum principles can be preserved. This result is based on the paper [3, Mincsovics, 2010].

   Consider the paragraph "Problem 2." then with the notations

$$m = \min_{\mathcal{T}_h} m_i, \quad M = \max_{\mathcal{T}_h} m_i, \quad G = \min_{\mathcal{T}_h} \cos\gamma_{ij},$$

$$\spadesuit = \frac{a_M}{2}\frac{(d+1)(d+2)}{m^2} + \frac{b_M}{2}\frac{d+2}{m} + c_M, \quad \heartsuit = a_m G\frac{(d+1)(d+2)}{M^2} - b_M\frac{d+2}{m} - c_M.$$

   the following theorem is valid.

Thm. 4.2.5. Let us assume that for the mesh $\mathcal{T}_h$ the geometrical-fineness condition

$$0 < \heartsuit \qquad \text{(mesh condition)}$$

holds.

Moreover the condition

$$\frac{\spadesuit}{\spadesuit + \heartsuit} \leq \theta \qquad \text{(restriction for the parameter } \theta\text{)}$$

holds, too. Then under the condition

$$\frac{1}{\theta}\frac{1}{\heartsuit} \leq \Delta t \leq \frac{1}{1-\theta}\frac{1}{\spadesuit} \qquad \text{(restriction for the time step } \Delta t\text{)}$$

$\mathcal{L}_{a,b,c}/\mathcal{L}_{a,b,0}$ satisfies the DmP/DMP.

In **Tab. 4.1. and 2.** We summarized the conditions obtained from Thm. 4.2.5. and the "real" conditions ensuring the DmP on some problem, respectively. With this we investigated the sharpness of the theorem.

4. We showed the relationship between discrete elliptic and parabolic maximum principles. These results are based on the paper [4, Mincsovics, 2010].

   In **Lem. 4.3.3.** We characterized the discrete stabilization property.

   Thm. 4.3.4. We proved that under the assumption that the hyper-matrix $\mathcal{L}$ possesses the DnP, the DnP of $\mathbf{K}$ is equivalent to the DSP of $\mathcal{L}$.

   Thm. 4.3.5. We proved that under the assumption that the hyper-matrix $\mathcal{L}$ defines a non-singular matrix $\mathbf{K}_0$, the DmP of $\mathcal{L}$ implies the DSP for $\mathcal{L}$, and the DwMP for $\mathbf{K}$.

   These results explain that a non-adequate mesh can already hinder to fulfil discrete parabolic maximum principles.

   Ex. 4.3.10. and 11. We gave numerical examples in order to illustrate Thm. 4.3.4.

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