# Study of Minimal Models of Chaotic Systems

Theses of the Ph.D. Dissertation

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## Introduction

A minimal model of a system is the simplest model which can describe its essential behavior qualitatively. The simplicity of the model can be achieved by the small numbers of elements and the simplicity of interactions between these elements. If, beyond a qualitatively correct description of the phenomenon, the model is also able to give good quantitative predictions, then its value is higher.

The aim of the Ph.D. dissertation is to construct minimal models of two physical systems and to understand the main properties of their behavior by these models. The systems are physically quite different from each other, but share dynamically very similar features. The first studied problem is chosen from the field of celestial mechanics and is motivated by the tidal effect in the Earth-Moon system. We would like to understand, among other things, why the same side of the Moon always faces the Earth. In our problem the moon, called secondary, is extended and can be deformed. The aim is, in general, to explore the connections between the deformations, the rotation, the change of the orbit, and the dissipation by the constructed model. The other studied problem is hydrodynamic, and the aim is to understand the motion of an inertial buoyant particle in a time-dependent vortex. The vortex is generated by a magnetic stirrer in a cylinder.

Both systems can be considered nonlinear and forced dynamical systems. In the first case, the rotating and deforming secondary is the system, and the forcing is determined by the orbit. In the second case, the flow is the forcing which influences the motion of the particle. In both cases, chaotic behavior is observable, and the systems affect to their own forcing, the rotation state of the secondary has an effect on the temporal change of the orbit, and the presence of the particle changes the original flow.

It can be stated about both systems that their study leads us to the fields of actual investigations. The spin-orbit problem can be studied in many aspects [A1, A2]. Constructions of a lot of simple models are connected to it. The numbers of theoretical and experimental investigations are, more or less, balanced in the case of heavy finite-size inertial particles [A3]. In the case of buoyant inertial particles, the number of experimental investigations is rather limited comparted to the theoretical one [A4]. Very little is known about the experimental side of the dynamics of these kind of particles. These facts increase the relevance of the experimental investigation of the hydrodynamic system.

## Methods

During our studies, the numerical simulations based on fourth order Runge—Kutta method [A6] dominate, but analytic calculations are widely used for completion as well. In the case of the hydrodynamic problem, experimental methods are also necessary to explore the properties of particle motion and the funnel of the flow, to check and tune parameters for simulations, and to prove the accuracy of the model. Motions are monitored by means of video recordings. After digitalization, it becomes possible to make time series of projections of motions. The effects of the turbulence in the flow [A7] are interpreted as noise [A5]. According to our approach, the noise is added to the motion in a relatively simple way by taking into account that turbulent vortices in the flow kick the particle and modify its velocity.

## Theses

1. **Construction of a model for the celestial mechanical problem and classification of the rotation states**

*a.* A dissipative model of the celestial mechanical problem has been constructed. Changes of the rotation states and the temporal change of the orbit can be examined in a consistent way by the model. The model contains an extended secondary represented by two point masses connected with a damped spring. The secondary (moon) orbits around the primary (planet) and its oscillation, and rotation are assumed to take place in the fixed plane of the orbit (see Fig. 1). The gravitational interactions of both point masses with the primary are taken into account, but those between the point masses are neglected. The model is the simplest model being able to describe the above mentioned phenomena in a natural way [T1, C1].

*b.* Rotation states of the secondary and the transitions between them have been classified. According to the classification, the rotation state can be permanent chaotic-like, transient chaotic, 1:1 resonance, and higher order resonances. The most important properties of the states are explored and reasons of the transitions are clarified.

*c.* We introduce a so-called resonance function to identify the resonant states. The resonance function is superior to the classical method based on the quotient of the orbital and rotational periods, as there are dynamically different resonant states which cannot be distinguished by the classical method but can be by the resonance function [T2, C5].

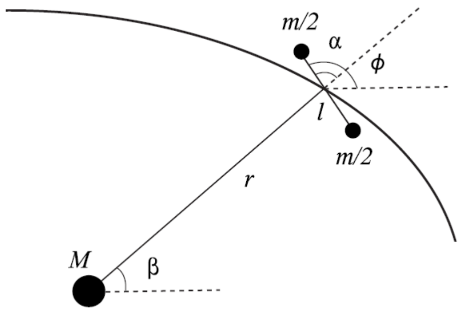


Fig. 1. Instantaneous configuration of the celestial mechanical system given by the generalized coordinates. Polar coordinates $r$ and $\beta$ are used for the center of mass of the secondary, with $l$ as the instantaneous length of the spring and $\phi$ the rotational angle characterizing the orientation of the secondary. The relative angle $\alpha$ is also indicated.

## 2. The 1:1 resonance

a. The 1:1 resonance plays a special role in the behavior of the system because the only attractor of the system is characterized by this resonant state. Additionally, in the attractor state the orbit is perfectly circularized [T2, C3].

b. In 1:1 resonance behavior of the system is qualitatively different from the behavior of other rotational states. This behavior can be easily described by the temporal change of the center-of-mass energy which decreases exponentially in time contrary to the linear behavior of the other states. We give the explanation of the exponential behavior and an analytical formula for the characteristic exponent [T2, C5].

c. According to our experience, the characteristic time of the decay of libration around the 1:1 resonant state is not a monotonic function of the strength of dissipation, if the orbit of the secondary is circular. We give an explanation of this phenomenon.

d. We demonstrate that the system can leave the 1:1 resonant state if the orbit is not perfectly circularized [T2]. We give the condition of that.

## 3. The higher order resonances

*a.* In these resonant states the temporal change of the center-of-mass energy is typically linear. We give the explanation of this phenomenon [T2].

*b.* We give a quantitative condition for the existence of a resonant state. The essence of this condition is that the resonant state ceases to exist if the center-of-mass system cannot pump enough energy to the system of the secondary [T2].

*c.* The study of the model enlightens the reasons of ceasing of the resonant states.

## 4. Chaotic behavior

As the model has only one nonchaotic attractor, the chaotic behavior of the system cannot be permanent, only transient. Despite this fact, due to the slow changes of the orbital elements, we can observe permanent chaotic-like behavior beside the transient chaotic one [T2].

*a.* We prove the presence of the permanent chaotic-like behavior by creating attractors with "frozen" orbital elements. The structural instabilities of the attractors are shown. Boundary crisis has been found as a possible reason of the sudden changes of the rotational state. We calculate the cumulative distribution function which characterizes the lifetime of the transient chaotic period after the crisis.

*b.* Presence of the transient chaotic behavior have been shown by studying transiently chaotic trajectories with different initial conditions and explained by the fractal property of the basin boundaries [T2].

## 5. Experimental results (hydrodynamic problem)

We have made measurements in order to construct a model flow about the vortex, and to make a model for describing the particle motion (see Fig. 2). We have analyzed the time series of the funnel and the particle motion. The dependencies of the particle and funnel motions on the frequency of the stirrer have been experimentally studied.

*a.* As a result of analyses of the funnel motion we can estimate main parameters of the flow and can make the conclusion that the flow has periodic properties.

*b.* As a result of analyses of the particle motion 1. we can estimate the velocity at the axis. We can make statistics about the particle positions and lifetimes. These are useful to tune model parameters. 2. We find that the vertical components of the successive velocity increments are clearly anticorrelated. 3. We analyze the noisy part of the particle motion by methods of statistics and find a height dependence which has important role in the right description of the statistical properties of the particle motion.

Based on the results of measurements, we have constructed a series of models of increasing complexity to provide an acceptable minimal model for the experimentally observed dynamics. At each stage, new features have been introduced [T3, C2].

## 6. The laminar model

We numerically simulated the constructed model. We study the particle motion numerically in a time-independent model flow and in a periodically time-dependent model flow.

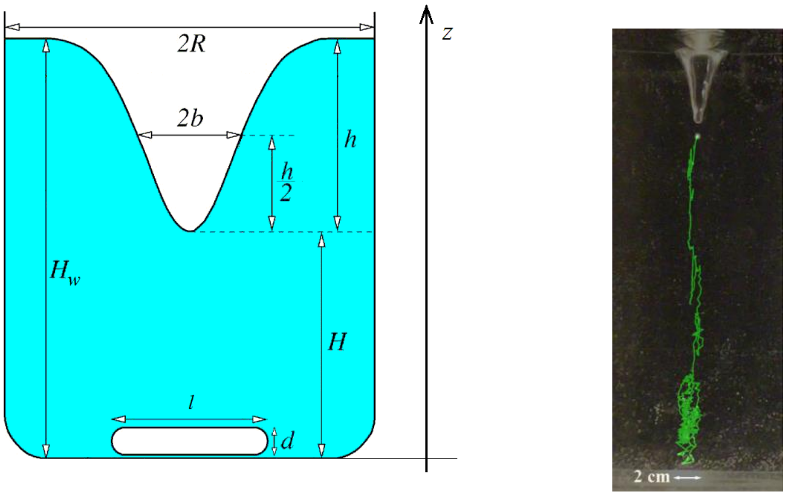


Fig. 2. Schematic diagram of the experimental setup (left) and the trajectory of a tracked particle (green line) from an experiment over 25 s (right). The geometrical parameters in the left panel are the radius of the cylinder $R$, the water height $H_w$, the height of the funnel $h$, and the half-width of the funnel $b$. $H = H_w - h$ is the distance between the deepest point of the funnel and the bottom of the container, and $l$ and $d$ are the length and diameter of the stirrer bar, respectively.

*a.* In the case of time-independent model flow: we determine the attractors and the conditions of their stability [T3, C2].

*b.* In the case of time-dependent model flow: 1. We show that in a time-dependent cylindrical symmetric flow both the permanent and the transient chaotic behavior are possible. Unfortunately, these chaotic behaviors are much more insignificant in the model than the experiments suggest and the typical behavior in the model is nonchaotic. 2. We admit that the experimentally studied chaotic-like particle trajectories qualitatively differ from the simulated ones and are not consequences of the periodic time dependence of the flow [T3, C4].

## 7. Consideration of the turbulence

According to the next stage of the model series the particle motion is influenced by the small-scale turbulent vortices, so it is necessary to add the effect of the turbulence to the model. We do that in a relatively simple way by taking into account that turbulent vortices in the flow kick the particle and suddenly modify its velocity with a vector $\Delta\mathbf{v}$. The kicks are instantaneous and the direction of the vector $\Delta\mathbf{v}$ of velocity increment is chosen randomly with uniform spatial distribution. The time periods between the kicks follow an exponential distribution (Poisson process). The magnitude of the vector $\Delta\mathbf{v}$ is chosen also randomly and its probability density function is assumed to be Gaussian with zero expected value. The standard deviation of the Gaussian is an empirical function of the height of the particle.

This kind of modification of the velocity can be represented as a noise is added to the particle motion. We tune the parameters of the noise by a part of the experimental results. There is no need to add the correlations directly it is enough to consider the averaged effect. In order to check the degree of agreements between the experiment and the simulation we use the experimental results which are not built in the model previously. Surprisingly, the agreements are good considering the simplicity of the model. [T3, C6, C7].

## Summary

In the dissertation, we investigate two systems which have similar dynamical properties but their physical structures are quite different. The most important features of their behaviors are clarified. By the study of the celestial mechanical problem the necessity of a new classification of the resonant states becomes clear. The new classification method is superior to the classical one as there are dynamically different resonant states which cannot be distinguished by the classical method but can by the new one. Rotation states of the secondary and the transitions between them are classified. The most important properties of the chaotic and resonant states are explored and reasons of the transitions are clarified.

About the hydrodynamic system, we finally succeed to construct an adequate minimal model. The minimal model is the last member of a model series. All members of the series introduce new features of the real system. It turns out that the essence of the particle motion can be understood without the direct numerical simulation of the Navier—Stokes equations.

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## Publication before the doctoral program


[E1] J. Vanyó and T. Tél (2007): *Dynamics of chaotic driving: Rotation in the restricted three-body problem*, Chaos **17**, 013113.


## Publications related to the theses


[T1] B. Escribano, J. Vanyó, I. Tuval, J. H. E. Cartwright, D. L. González, O. Piro, and T. Tél (2008): *Dynamics of tidal synchronization and orbit circularization of celestial bodies*, Phys. Rev. E **78**, 036216.

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## Conference talks and posters related to the theses

[C1] J. Vanyó, B. Escribano, I. Tuval, J. H. E. Cartwright, D. L. González, O. Piro, and T. Tél (2008): *Dynamics of tidal synchronization and orbit circularization of celestial bodies* (oral presentation), Dynamics Days 2008, Delft, Netherlands.

[C2] J. Vanyó, I. M. Jánosi, and T. Tél (2008): *Chaotic motion of light particles in an unsteady three dimensional vortex: experiments and simulation* (poster), Dynamics of Inertial Particles: From Ocean and Atmosphere to Planets (DIPOAP 2008), Drezda, Germany.

[C3] J. Vanyó, B. Escribano, J. H. E. Cartwright, D. L. González, O. Piro, and T. Tél (2009): *Toy model for tidal synchronization and orbit circularization* (poster), The Fifth International Meeting on Celestial Mechanics (CELMEC V), Viterbo, Italy.

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[C7] J. Vanyó, M. Vincze, I. M. Jánosi, and T. Tél (2014): *Chaotic motion of light particles in an unsteady three dimensional vortex: experiments and simulation* (oral presentation), Experimental Chaos and Complexity (ECC2014), Aberdeen, United Kingdom.