Theses

of the PhD Thesis

Characterizations, extensions, and factorizations of Hilbert space operators

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PhD Thesis

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Our aim in the present dissertation is to discuss some questions of the operator theory of Hilbert spaces. More precisely, we investigate characterization, factorization, and extension problems of various classes of operators and suboperators, respectively. The dissertation is based on the author's papers [**1, 2, 5, 6, 7, 8**]. The results of Chapter 1 and 2 are taken from the manuscripts [**3, 4**] of the author.

1. Characterization of selfadjoint operators

Revision of von Neumann's characterization of selfadjoint operators among symmetric operators and its applications is our main purpose. Algebraic arguments we use goes back to Arens, contrary to the geometric nature of Cayley transform used by von Neumann. We do not assume that the underlying Hilbert space is complex, nor that the corresponding symmetric operator is densely defined: it is a consequence.

The Hilbert spaces $\mathfrak{H}, \mathfrak{K}$ are real or complex (or quaternionic) and the operators A, B acting between them are not necessarily densely defined.

Theorem 1.1. *Let $A : \mathfrak{H} \to \mathfrak{H}$ be a linear operator. The following statements are equivalent:*

(i) *A is selfadjoint.*
(ii) a) *A is symmetric,*
b) $\ker A = \{\operatorname{ran} A\}^{\perp}$,
c) $\operatorname{ran} A = \big\{y \in \mathfrak{H} : \sup\{|(x,y)|^2 : x \in \operatorname{dom} A, (Ax, Ax) \leq 1\} < \infty\big\}$.

Selfadjointness of T^*T was proved by von Neumann in case of closed densely defined operator T between complex Hilbert spaces. Without the closedness assumption we have the following statement.

Theorem 1.2. *Let $T : \mathfrak{H} \to \mathfrak{K}$ be a densely defined operator between real or complex Hilbert spaces. The following statements are equivalent:*

(i) *T^*T is selfadjoint.*
(ii) $\operatorname{ran}(I + T^*T) = \mathfrak{H}$.

Our revised version of von Neumann's characterization of skew-adjoint operators on real or complex Hilbert space follow by the next result:

Theorem 1.3. *Let A be a not necessarily densely defined linear operator in a real or complex Hilbert space $\mathfrak{H}$. Then equivalent statements are:*

(i) *A is skew-selfadjoint.*
(ii) a) *A is skew-symmetric,*
b) $\operatorname{ran}(I + A) = \mathfrak{H}$,
c) $\operatorname{ran}(I - A) = \mathfrak{H}$.

Full range property on the product space of an appropriate square matrix operator characterizes the selfadjointness of a not necessarily densely defined symmetric operator. We mention here that by using of this result we are allowed to discuss the real and the complex variants of the Kato–Rellich perturbation theorem jointly.

Theorem 1.4. *Let A be a not necessarily densely defined operator in a real or complex Hilbert space $\mathfrak{H}$. The following statements are equivalent:*

(i) *A is selfadjoint.*

(ii) a) *A is symmetric,*

b) $\operatorname{ran}\begin{pmatrix} A & I \\ -I & A \end{pmatrix} = \mathfrak{H} \times \mathfrak{H}$.

The content of this chapter can be found in [**3**].

2. Characterization of essentially selfadjoint operators

A symmetric/skew-symmetric operator A on a real or complex Hilbert space is called *essentially selfadjoint/skew-adjoint* if its adjoint A^* exists (i.e. A is densely defined) and A^* is selfadjoint/skew-adjoint, respectively. In other words, A^* fulfills property $A^* = \pm A^{**}$, where A^{**} is nothing else that the closure of A. In what follows we give a simple range-characterization for the case just mentioned.

Theorem 2.1. *Let A be densely defined symmetric/skew-symmetric operator on a real or complex Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is essentially selfadjoint/skew-adjoint;*

(ii) $\operatorname{ran} A^* = \operatorname{ran} A^{**}$*;*

(iii) $\pm A^* \subset A^{**}$*, i.e. A^* is symmetric/skew-symmetric.*

In what follows a (skew-)symmetric operator A is given on a real or complex Hilbert space, not at all assuming that it is densely defined, but under some natural conditions it turns out to be even essentially skew/self-adjoint:

Theorem 2.2. *Let $\mathfrak{H}$ be real or complex Hilbert space, $A : \mathfrak{H} \to \mathfrak{H}$ be closable not necessarily densely defined and (skew-)symmetric linear operator. The following two statements are equivalent:*

(i) *A is essentially skew/self-adjoint;*

(ii) a) *A is (skew-)symmetric*

b) $\{\operatorname{ran} A\}^\perp = \{x \in \mathfrak{H} : \exists \{x_n\}_{n=1}^\infty \subset \operatorname{dom} A, x_n \to x, Ax_n \to 0\}$

c) *The following linear manifolds*

$$\begin{cases} \mathscr{R}_* := \big\{y \in \mathfrak{H} : \sup\{|(x,y)|^2 : x \in \operatorname{dom} A, (Ax, Ax) \le 1\} < \infty\big\}, \\ \mathscr{R}_{**} := \big\{z \in \mathfrak{H} : \exists \{x_n\}_{n=1}^\infty \subset \operatorname{dom} A \textit{ such that } Ax_n \to z, (x_n - x_m) \to 0\big\} \end{cases}$$

are equal.

An essentially skew-adjoint operator on real or complex Hilbert space is basically simple to characterize among the non-densely defined skew-symmetric closable operators by the Neumann type condition.

Theorem 2.3. *Let $\mathfrak{H}$ be real or complex Hilbert space and A be a linear skew-symmetric closable operator not assumed to be densely defined. The following statements are equivalent:*

(i) $A^* = -\overline{A}$, *i.e. A is densely defined and essentially skew-adjoint;*
(ii) $\{\mathrm{ran}(I + A)\}^\perp = \{\mathrm{ran}(I - A)\}^\perp = \{0\}$.

Characterization of positive essentially selfadjoint operators is given in the following result. Note that we do not assume that the operator A is densely defined, only that the operator A is symmetric (what is automatic if the space is complex).

Theorem 2.4. *Let A be a positive symmetric closable linear operator on the real or complex Hilbert space $\mathfrak{H}$. Then equivalent statements are:*

(i) *A is densely defined and essentially selfadjoint;*
(ii) $\{\mathrm{ran}(I + A)\}^\perp = \{0\}$.

Finally we give necessary and sufficient conditions in terms of dense range property of an appropriate square matrix operator in the product space $\mathfrak{H} \times \mathfrak{H}$:

Theorem 2.5. *Let $\mathfrak{H}$ be real or complex Hilbert space, $A : \mathfrak{H} \to \mathfrak{H}$ not necessarily densely defined operator. The following statements are equivalent:*

(i) *A is densely defined and essentially selfadjoint;*
(ii) a) *A is closable;*
b) *A is symmetric;*
c) $\mathrm{ran}\begin{pmatrix} A & I \\ -I & A \end{pmatrix}$ *is dense in $\mathfrak{H} \times \mathfrak{H}$.*

The content of this chapter is based on [**4**].

3. T^*T always has a positive selfadjoint extension

One of the basic results of the theory of unbounded operators is due to John von Neumann: if T is a densely defined closed operator between Hilbert spaces then T^*T is a positive selfadjoint operator. If we leave out the condition that T is closed, the conclusion is not true anymore. In the present chapter, following and simplifying the treatment of Sebestyén and Stochel, we prove that T^*T always admits the Krein–von Neumann extension via von Neumann's theorem using a model Hilbert space, the range closure of the restriction of T to the domain of T^*T.

Theorem 3.1. *Let T be a densely defined operator between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. Then T^*T has a positive selfadjoint extension.*

Necessary and sufficient conditions for the selfadjointness of T^*T are given in the next result:

Theorem 3.2. *Let T be a densely defined operator between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. The operator T^*T in the Hilbert space $\mathfrak{H}$ is selfadjoint if and only if one of the following conditions holds:*

$$\operatorname{dom} T^*T = \big\{g \in \mathscr{D}_* : \exists h \in \mathscr{R}_*, (T^*Tf, g) = (f, h) \text{ for all } f \in \operatorname{dom} T^*T\big\}. \tag{3.1}$$

$$\begin{cases} \ker T^*T = \big(\operatorname{ran} T^*T\big)^{\perp}, \\ \operatorname{ran} T^*T = \big\{h \in \operatorname{ran} T_* : \sup\{|(f,h)| : f \in \operatorname{dom} T^*T, \|T^*Tf\| \le 1\} < \infty\big\}, \end{cases} \tag{3.2}$$

where T_* stands for the closure of the restriction of T^* to $\operatorname{dom} T^* \cap \operatorname{ran} T$.

Finally, the following theorem describes all densely defined positive operators and their Friedrichs extensions:

Theorem 3.3. *If A is a densely defined positive operator in a Hilbert space $\mathfrak{H}$ then there is densely defined closable operator T acting in $\mathfrak{H}$ such that*

$$A = T^*T$$

and that T^*T^{**} is the Friedrichs extension of A.

This chapter contains the results of [**2**].

4. Operator extensions with closed range

Throughout this chapter our basic tool is the factorization argument of Z. Sebestyén and J. Stochel. Their method plays a central role also in the remaining parts of this dissertation, we therefore recall it here.

Let A be a positive operator in the Hilbert space $\mathfrak{H}$ such that

$$\mathscr{D}_*(A) := \{k \in \mathfrak{H} : |(Ah, k)|^2 \le m_k \cdot (Ah, h) \text{ for all } h \in \operatorname{dom} A\}$$

Equip the range space $\operatorname{ran} A$ with the inner product $\langle \cdot, \cdot \rangle_A$ defined by

$$\langle Ah, Ak \rangle_A = (Ah, k) \qquad (h, k \in \operatorname{dom} A).$$

Let $\mathfrak{H}_A$ denote the completion of this pre-Hilbert space. The natural embedding operator J_A of $\operatorname{ran} A \subseteq \mathfrak{H}_A$ into $\mathfrak{H}$ defined by $J_A(Ah) := Ah$ for $h \in \operatorname{dom} A$ is then densely defined between $\mathfrak{H}_A$ and $\mathfrak{H}$ so that its adjoint J_A^* has dense domain $\operatorname{dom} J_A^* = \mathscr{D}_*(A)$. According to the Schwarz-inequality one obtains that $\operatorname{dom} A \subseteq \mathscr{D}_*(A)$ and $J_A^* x = Ax \in \mathfrak{H}_A$ for $x \in \operatorname{dom} A$. According to Neumann's theorem, $J_A^{**} J_A^*$ is a positive selfadjoint extension of A, and it turns out that $J_A^{**} J_A^*$ is just the Krein–von Neumann extension of A, i.e. the smallest among all positive selfadjoint extensions of A.

The main theorem of this chapter is the following result characterizing those operators which have a positive selfadjoint extension with closed range:

Theorem 4.1. *Let A be a (not necessarily densely defined or closed) positive operator in the Hilbert space $\mathfrak{H}$ such that the linear manifold $\mathscr{D}_*(A)$ is dense, i.e. the Krein–von Neumann extension of A exists. The following statements are equivalent:*

(i) *There exists a constant $m_A \geq 0$ such that for each $h \in \operatorname{dom} A$ the following inequality holds:*

$$(Ah, h) \leq m_A \cdot \|Ah\|^2. \tag{4.1}$$

(ii) *The Krein–von Neumann extension of A has closed range.*

(iii) *There is a positive selfadjoint extension $\widetilde{A}$ of A that has closed range.*

The Moore–Penrose pseudoinverse of the Krein–von Neumann extension can also be given via factorization as follows:

Theorem 4.2. *Let A be a positive operator such that A admits its Krein–von Neumann extension A_N which has closed range. Then the operator V_A acting between $\mathfrak{H}$ and $\mathfrak{H}_A$ defined by the following relation*

$$\operatorname{dom} V_A = \{Ah + k : h \in \operatorname{dom} A, k \in \operatorname{ran} A^\perp\}, \qquad V_A(Ah + k) = Ah \in \mathfrak{H}_A$$

is densely defined and bounded such that the Moore-Penrose pseudoinverse $(A_N)^\dagger$ of A_N equals $V_A^* V_A^{**}$ and has the norm

$$\|(A_N)^\dagger\| = \inf\{\gamma \geq 0 : (Ah, h) \leq \gamma \cdot \|Ah\|^2 \ \textit{for all}\ h \in \operatorname{dom} A\}. \tag{4.2}$$

In the last result of this chapter some characterizations of essentially selfadjointness of positive operators are presented. The notions $(A + I)_N$ and $(A + I)_F$ below stand for the Krein–von Neumann and the Friedrichs extensions of $A + I$, respectively.

Theorem 4.3. *Let A be a densely defined positive operator in a Hilbert space $\mathfrak{H}$. The following statements are equivalent:*

(i) *A is essentially selfadjoint;*

(ii) *$A + I$ admits a unique positive selfadjoint extension, i.e. $(A + I)_N = (A + I)_F$;*

(iii) *The range of $A + I$ is dense in $\mathfrak{H}$;*

(iv) *$(A + I)_N = A_N + I$;*

(v) *$(A + I)_N$ has bounded inverse.*

5. On form sums of positive operators

If A is a positive selfadjoint operator in a Hilbert space $\mathfrak{H}$ then the following relation

$$\mathfrak{t}_A(f, g) := (A^{1/2} f, A^{1/2} g), \qquad f, g \in \operatorname{dom} A^{1/2}, \tag{5.1}$$

defines a positive definite closed sesquilinear form over $\operatorname{dom} A^{1/2}$. If another positive selfadjoint operator B in $\mathfrak{H}$ is given such that $\operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$ is dense, then the representation theorem yields a unique positive selfadjoint operator C with the characteristic

properties $\operatorname{dom} C \subset \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$ and

$$(Cf, g) = (A^{1/2}f, A^{1/2}g) + (B^{1/2}f, B^{1/2}g),$$

for $f \in \operatorname{dom} C, g \in \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$. C is used to be called the *form sum* of the positive operators A and B and is usually denoted by $A \dotplus B$.

The following theorem gives an alternative construction for the form sum of two positive selfadoint operators. We refer to the notions of the previous section.

Theorem 5.1. *Let A and B be positive operators such that $\mathscr{D}_*(A) \cap \mathscr{D}_*(B)$ is dense. Let $\mathfrak{H}_A, \mathfrak{H}_B$ and J_A, J_B denote the appropriate auxiliary Hilbert spaces and operators. Let $J : \mathfrak{H}_A \times \mathfrak{H}_B \to \mathfrak{H}$ be stand for the following linear operator*

$$\{Af, Bg\} \mapsto Af + Bg, \qquad f \in \operatorname{dom} A, g \in \operatorname{dom} B. \tag{5.2}$$

Then the form sum of A_N and B_N equals $J^{**}J^*$.

The construction above makes it possible to examine the domain, the kernel and the range of the form sum and its square root:

Theorem 5.2. *Let A and B be positive selfadjoint operators in the Hilbert space $\mathfrak{H}$ such that $\operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$ is dense. Then*

(a) $\operatorname{dom}(A \dotplus B)^{1/2} = \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$.

(b) $\ker(A \dotplus B)^{1/2} = \ker(A \dotplus B) = \{\operatorname{ran} A + \operatorname{ran} B\}^{\perp}$.

(c) *For any $y \in \mathfrak{H}$ the following statements are equivalent:*

  (i) $y \in \operatorname{ran}(A \dotplus B)^{1/2}$;

  (ii) *There is a nonnegative constant m_y such that for any $h \in \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$*

$$|(y, h)|^2 \leq m_y \left(\|A^{1/2}h\|^2 + \|B^{1/2}h\|^2\right).$$

  (iii) *There are two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ from $\operatorname{dom} A$ and $\operatorname{dom} B$, respectively, such that $\{A^{1/2}f_n\}_{n=1}^{\infty}$ and $\{B^{1/2}g_n\}_{n=1}^{\infty}$ are convergent and that $Af_n + Bg_n \to y$.*

(d) *For any $h \in \mathfrak{H}$ the following statements are equivalent*

  (i) $h \in \operatorname{dom}(A \dotplus B)$;

  (ii) $h \in \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2}$ *and there are two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ from $\operatorname{dom} A$ and $\operatorname{dom} B$, respectively, with $A^{1/2}f_n \to A^{1/2}h$, $B^{1/2}g_n \to B^{1/2}h$ such that $\{Af_n + Bg_n\}_{n=1}^{\infty}$ converges.*

(e) *For any $y \in \mathfrak{H}$ the following statements are equivalent*

  (i) $h \in \operatorname{ran}(A \dotplus B)$;

  (ii) *There are two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ from $\operatorname{dom} A$ and $\operatorname{dom} B$, respectively, such that $\{A^{1/2}f_n\}_{n=1}^{\infty}$ and $\{B^{1/2}g_n\}_{n=1}^{\infty}$ are convergent, $Af_n + Bg_n \to y$*

and that for any $f \in \operatorname{dom} A$ *and* $g \in \operatorname{dom} B$

$$\lim_{n\to\infty} \left|(A^{1/2}f_n, A^{1/2}f) + (B^{1/2}g_n, B^{1/2}g)\right| \leq m_y\|Af + Bg\|,$$

where m_y *is a nonnegative constant depending only on* y.

Our construction and the results of the previous chapter can also be applied for characterizing those pairs of positive operators whose form sum has closed range:

Theorem 5.3. *Let* A *and* B *be positive operators in the Hilbert space* $\mathfrak{H}$ *such that* $\mathscr{D}_*(A)\cap \mathscr{D}_*(B)$ *is dense. The form sum of* A_N *and* B_N *has closed range if and only if there is constant* $\gamma > 0$ *such that*

$$M_A(h) + M_B(h) \geq \gamma \cdot \|h\|^2$$

for each $h \in \mathscr{D}_*(A) \cap \mathscr{D}_*(B) \cap \overline{\operatorname{ran} A + \operatorname{ran} B}$, *where* $M_A(h)$ *and* $M_B(h)$ *are defined by*

$$\begin{cases} M_A(h) := \sup\left\{|(Af, h)|^2 : f \in \operatorname{dom} A, (Af, f) \leq 1\right\}, \\ M_B(h) := \sup\left\{|(Bg, h)|^2 : g \in \operatorname{dom} B, (Bg, g) \leq 1\right\}. \end{cases}$$

If this is the case then

$$\operatorname{ran}(A_N \dotplus B_N) = \overline{\operatorname{ran} A + \operatorname{ran} B}.$$

The Moore–Penrose pseudoinverse is also determined in the next result:

Theorem 5.4. *Let* A *and* B *be positive operators in the Hilbert space* $\mathfrak{H}$ *such that* $\mathscr{D}_*(A)\cap \mathscr{D}_*(B)$ *is dense and that* $A_N \dotplus B_N$ *has closed range. Then we have*

$$\operatorname{ran} A + \operatorname{ran} B \subseteq \operatorname{ran}(A_N \dotplus B_N),$$

i.e. for any $f \in \operatorname{dom} A$ *and* $g \in \operatorname{dom} B$ *there is a* $h \in \operatorname{dom} J^{**}J^*$ *with* $J^{**}J^*h = Af + Bg$. *If* V *denotes the mapping from* $\mathfrak{H}$ *into* $\mathfrak{H}_A \times \mathfrak{H}_B$ *defined by the following correspondence*

$$\operatorname{dom} V = \left\{Af + Bg + k : f \in \operatorname{dom} A,\ g \in \operatorname{dom} B, k \in \{\operatorname{ran} A + \operatorname{ran} B\}^{\perp}\right\},$$
$$Af + Bg + k \mapsto J^*h,$$

then V *is a densely defined bounded linear operator such that* $V^{**}V^* = \left(A_N \dotplus B_N\right)^{\dagger}$, *that is* $V^{**}V^*$ *equals the Moore-Penrose pseudoinverse of* $A_N \dotplus B_N$. *Furthermore, the norm* $\left\|\left(A_N \dotplus B_N\right)^{\dagger}\right\|$ *can be computed as the following infimum:*

$$\inf\left\{\gamma \geq 0 : \|k\|^2 \leq \gamma(M_A(k) + M_B(k))\ \textit{for all}\ k \in \mathscr{D}_*(A) \cap \mathscr{D}_*(B) \cap \overline{\operatorname{ran} A + \operatorname{ran} B}\right\},$$

where $M_A(k)$ *and* $M_B(k)$ *are defined in the previous theorem.*

6. Biorthogonal expansions for symmetrizable operators

Let A and B be bounded operators in a Hilbert space $\mathfrak{H}$ such that A is positive. Assume furthermore that the identity

$$AB = B^*A$$

is satisfied. This fact is also expressed by saying that B is *symmetrizable* with respect to A (on the left). The main theorems of this chapter generalizing some results of Krein are the following:

Theorem 6.1. *Let B be a compact symmetrizable operator. Then there exists a biorthogonal sequence of vectors $\{e_n, f_n\}_{n\in\mathbb{N}}$ and a sequence of nonnegative real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ such that the following spectral expansions hold*

$$PBx = \sum_{k=1}^{\infty} \lambda_k (x, f_k) e_k \qquad (x \in \mathfrak{H}),$$

$$B^*Px = \sum_{k=1}^{\infty} \lambda_k (x, e_k) f_k \qquad (x \in \mathfrak{H}),$$

where the convergence is in the norm of the Hilbert space and P stands for the orthogonal projection of $\mathfrak{H}$ onto $\overline{\operatorname{ran} A}$.

Theorem 6.2. *Let T be a selfadjoint operator such that $TA^{1/2}$ is compact. Then there exists an A-orthonormal sequence $\{e_k\}_{k\in\mathbb{N}}$ in the sense $(Ae_k, e_l) = \delta_l^k$ for $k, l \in \mathbb{N}$ and a sequence of real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ which converges to zero such that the spectral expansion satisfies*

$$PTPx = \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k \qquad (x \in \mathfrak{H})$$

where P is the orthonormal projection onto $\overline{\operatorname{ran} A}$ and the convergence for PTP is uniform in norm on the unit ball.

7. Lebesgue-type decomposition of positive operators

Let A and B be bounded positive operators acting on the complex Hilbert spaces $\mathfrak{H}$. Then B is called *absolutely continuous* (or in other words *closable*) with respect to A if for any sequence $\{x_n\}_{n=1}^{\infty}$ from $\mathfrak{H}$ with $(Ax_n, x_n) \to 0$ and $(B(x_n - x_m), x_n - x_m) \to 0$ yields $(Bx_n, x_n) \to 0$. Furthermore, B is *singular* with respect to A if for any positive linear operator C the properties $C \le A$ and $C \le B$ imply $C = 0$.

The main result of this chapter is a revised version of Ando's appropriate result giving Lebesgue-type decomposition of the positive operator B with respect to A:

Theorem 7.1. *Let A and B be bounded positive linear operators on a complex Hilbert space $\mathfrak{H}$. Then*

$$\mathfrak{M} = \left\{\xi \in \mathfrak{H}_B : \exists \{x_n\}_{n=1}^{\infty} \text{ in } \mathfrak{H}, (Ax_n, x_n) \to 0, Bx_n \to \xi\right\}$$

is a closed linear subspace of $\mathfrak{H}_B$ and if P stands for the orthogonal projection of $\mathfrak{H}_B$ onto $\mathfrak{M}$ then we have the following decomposition for B:

$$B = B_c + B_s,$$

where $B_c = J_B^{**}(I-P)J_B^*$ is absolutely continuous with respect to A, and $B_s = J_B^{**}PJ_B^*$ is singular with respect to A. Moreover, if C is a positive linear operator such that C is absolutely continuous with respect to A and $C \leq B$ then $C \leq J_B^{**}(I-P)J_B^* = B_c$.

By defining the following linear relation on $\mathfrak{H}_A \times \mathfrak{H}_B$

$$\widehat{B} := \left\{\{Ax, Bx\} \in \mathfrak{H}_A \times \mathfrak{H}_B : x \in \mathfrak{H}\right\},$$

the following characterization of absolute continuity can be presented:

Theorem 7.2. *Let $A, B \in \mathscr{B}(\mathfrak{H})$ be positive linear operators. The following statements are equivalent:*

(i) *B is absolutely continuous with respect to A.*
(ii) *$\mathfrak{M} = \{0\}$.*
(iii) *The linear relation $\widehat{B}$ above defines a closable operator from $\mathfrak{H}_A$ into $\mathfrak{H}_B$.*
(iv) *The set $\left\{\xi \in \mathfrak{H}_B : J_B^{**}\xi \in \operatorname{ran} A^{1/2}\right\}$ is dense in $\mathfrak{H}_B$.*
(v) *The set $\left\{x \in \mathfrak{H} : B^{1/2}x \in \operatorname{ran} A^{1/2}\right\}$ is dense in $\mathfrak{H}$.*

If any of the above conditions is satisfied then we have the following factorization for B:

$$B = (\widehat{B}J_A^*)^*(\widehat{B}J_A^*).$$

In the next theorem we characterize the A-singular positive operators:

Theorem 7.3. *Let $A, B \in \mathscr{B}(\mathfrak{H})$ be positive operators. The following statements are equivalent:*

(i) *B is singular with respect to A.*
(ii) *$\mathfrak{M} = \mathfrak{H}_B$.*
(iii) *The linear relation $\widehat{B}$ is maximally singular, i.e. $\operatorname{dom} \widehat{B}^* = \{0\}$.*
(iv) *$J_B^{**}\xi \in \operatorname{ran} A^{1/2}$ for some $\xi \in \mathfrak{H}_B$ implies $\xi = 0$.*
(v) $\operatorname{ran} A^{1/2} \cap \operatorname{ran} B^{1/2} = \{0\}$.

Contrary to the case of measures, the Lebesgue-type decomposition among positive operators is not necessarily unique. However, a necessary and sufficient condition guaranteeing uniqueness in the decomposition can be provided:

Theorem 7.4. *Let A and B be positive linear operators in the Hilbert space $\mathfrak{H}$. The following assertions are equivalent:*

(i) *The Lebesgue-type decomposition of B into A-absolute continuous and A-singular parts is unique.*

(ii) $\operatorname{dom} \widehat{B}^* \subseteq \mathfrak{H}_B$ *is closed.*

(iii) $J_B^{**}\big(\mathfrak{M}^\perp\big) \subseteq \operatorname{ran} A^{1/2}$.

(iv) *The regular part of $\widehat{B}$, i.e. the linear operator $\mathfrak{H}_A \supseteq \operatorname{ran} A \to \mathfrak{H}_B$ defined by the correspondence $Ax \mapsto (I-P)Bx$ is continuous.*

(v) $B_c := J_B^{**}(I-P)J_B^*$ *is dominated by A, i.e. $B_c \leq \alpha A$ for some nonnegative constant α.*

The results of this chapter are taken from [**7**]

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