On the Intersection of Geodesic Balls

Theses of the PhD Doctoral Dissertation

by

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1 Introduction

The motivation of this dissertation was the Kneser–Poulsen conjecture. In 1954 Ebbe Thue Poulsen [19] and in 1955 Martin Kneser [14] independently formulated the conjecture that if a finite collection of congruent balls in Euclidean space is rearranged in such a way that the distances between the centers decrease, then the volume of the union of the balls does not increase, that is,

Conjecture 1.1. *If the points $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ lie in the Euclidean space $\mathbb{E}^n$ and the inequalities $d(P_i, P_j) \geq d(Q_i, Q_j)$ are satisfied for all $1 \leq i < j \leq k$, then for arbitrary $r > 0$, we have*

$$\mathrm{Vol}_n\left(\bigcup_{i=1}^{k} \mathcal{B}(P_i, r)\right) \geq \mathrm{Vol}_n\left(\bigcup_{i=1}^{k} \mathcal{B}(Q_i, r)\right),$$

where Vol_n denotes the n-dimensional volume and $\mathcal{B}(P, r)$ stands for the ball of radius r centered at P.

An important special case is when the balls are moved continuously, i.e., when there are continuous functions $\gamma_i \colon [0,1] \to \mathbb{E}^n$ for $i = 1, \dots, k$ such that $\gamma_i(0) = P_i$ and $\gamma_i(1) = Q_i$ and $\|\gamma_i - \gamma_j\|$ is monotonically decreasing function for all $1 \leq i, j \leq k$. For $n = 2$, this special case was proved by Béla Bollobás in 1968 ([3]). In 1955 Balázs Csikós ([5]) generalized this result for disks with different radii (when the radius r can depend on i). In the paper [6], Balázs Csikós extended this statement to arbitrary dimension. These results motivate the following conjecture.

Conjecture 1.2. *If $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ are points in the Euclidean space $\mathbb{E}^n$ such that the inequalities $d(P_i, P_j) \geq d(Q_i, Q_j)$ are satisfied for all $1 \leq i < j \leq k$, then for arbitrary positive numbers $r_1, \dots, r_k$, we have*

$$\mathrm{Vol}_n\left(\bigcup_{i=1}^{k} \mathcal{B}(P_i, r_i)\right) \geq \mathrm{Vol}_n\left(\bigcup_{i=1}^{k} \mathcal{B}(Q_i, r_i)\right).$$

In 1999 Balázs Csikós proved Conjecture 1.2 in the case when the balls are pushed together continuously ([7]) not only in Euclidean space, but also in spherical and hyperbolic spaces. In 2001 Robert Connelly and Károly Bezdek proved Conjecture 1.2 for $n = 2$ without any other restriction in [2]. In 2006 Balázs Csikós sharpened the result of his above mentioned paper [7] by proving that it is enough to suppose the existence of the continuous contraction in the space of dimension $(n + 2)$ ([8]). One of the two key formulas used in the proof is valid also in Einstein manifolds.

As the conjecture makes sense in any Riemannian manifold, one can ask whether the conjecture can be true in Riemannian manifolds more general than the constant curvature spaces. In 1996 Balázs Csikós and Gábor Moussong showed in [13] that the conjecture is not true in the elliptic space even in the continuous motion case. Finally, in 2010 Balázs Csikós and Dávid Kunszenti-Kovács proved that Conjecture 1.2 is not extensible to Riemannian manifolds more general than the spaces of constant curvature spaces. In the dissertation, we show that Conjecture 1.1 is not extensible either.

If the Kneser–Poulsen conjecture is true in a Riemannian manifold, then it is also true that the volume of the union of balls depends only on the radii of the balls and the distances between the centers. We can replace the union with the intersection by the following statement, which is a consequence of the inclusion-exclusion principle.

Proposition 1.3. *In a Riemannian manifold, the volume of the union of k geodesic balls depends only on the radii of the balls and the distances between the centers if and only if the analogous statement is true for the intersections.*

A similar statement is true for balls with equal radii. These claims motivate the introduction of the KP_k and $KP_k^=$ properties, examined in the dissertation.

Definition 1.4. We say that a Riemannian manifold has the KP_k *property* ($k \in \mathbb{Z}_+$) if the volume of the intersection of k geodesic balls depends only on the radii of the balls and the distances between the centers.

By Proposition 1.3, if the Kneser–Poulsen conjecture for balls with different radii is true in a Riemannian manifold, then the manifold has the KP_k property for each $k \in \mathbb{Z}_+$. We can introduce a weaker property according to the original conjecture.

Definition 1.5. A Riemannian manifold has the $KP_k^=$ *property* ($k \in \mathbb{Z}_+$) if the volume of the intersection of k geodesic balls with equal radii depends only on the common radius of the balls and the distances between the centers.

As a summary, the following implications are true.

$$\begin{array}{ccccc}
\begin{array}{c}\text{Kneser–Poulsen conjecture}\\ \text{for } k \text{ balls with different radii}\end{array} & \Rightarrow & KP_k & \Rightarrow & KP_{k-1} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\begin{array}{c}\text{Kneser–Poulsen conjecture}\\ \text{for } k \text{ balls with equal radii}\end{array} & \Rightarrow & KP_k^= & \Rightarrow & KP_{k-1}^=.
\end{array}$$

The main purpose of the dissertation is to characterize Riemannian manifolds having these properties.

The KP_1 and $KP_1^=$ properties are obviously the same, and they just mean that the volume of a geodesic ball depends only on the radius of the ball. It is known that these manifolds are of constant scalar curvature (in the case $n = 2$, this implies that the space is of constant curvature). The KP_1 property is closely related to the notion of ball-homogeneity, introduced by Oldřich Kowalski and Lieven Vanhecke in [15]. A Riemannian manifold is called ball-homogeneous if the volume of "small" geodesic balls depends only on the radius.

One of the two main chapters of the dissertation is about manifolds having the KP_2 or $KP_2^=$ property, the other one is about manifolds with the $KP_3^=$ property.

A manifold having the KP_k or $KP_k^=$ property for $k \geq 3$ also has the $KP_3^=$ property. We will see that if a $KP_3^=$ manifold is connected and complete, then this manifold is one of the simply connected spaces of constant curvature. These spaces are known to have the KP_k and $KP_k^=$ properties for all $k \in \mathbb{Z}_+$.

2 The intersection of two balls

In the second chapter of the dissertation, we study Riemannian manifolds having the KP_2 or $KP_2^=$ property. The presented results are based on the papers [9] and [10].

2-point homogeneous spaces obviously have these properties. Recall that a connected Riemannian manifold M is called 2-point homogeneous if for any 4 points $P, Q, P', Q' \in M$ such that $d(P,Q) = d(P',Q')$, there is an isometry $f\colon M \to M$ such that $f(P) = P'$ and $f(Q) = Q'$. The Lichnerowicz conjecture predicted that harmonic manifolds are 2-point homogeneous. Harmonic spaces have several equivalent characterizations, in the dissertation, we use the definition requiring that small geodesic spheres have constant mean curvature. Zoltán I. Szabó proved in [20] that if the universal covering space of the manifold is compact, then the Lichnerowicz conjecture is true. Later Ewa Damek and Fulvio Ricci constructed harmonic spaces which are not 2-point homogeneous (these are the Damek–Ricci spaces).

Zoltán I. Szabó also showed ([20]) that the connected, simply connected, complete harmonic spaces have the KP_2 property. Our goal was to prove the reversed statement, saying that every manifold having the KP_2 property is harmonic. For this purpose, we use the following asymptotical formula for the volume of two slightly intersecting regular domains.

Theorem 2.1 ([9])**.** *In an n-dimensional Riemannian manifold M, consider the regular domains D_1 and D_2 which are tangent to each other at a unique point P. Suppose that D_1 is compact and denote by Σ_1 and Σ_2 the boundary of D_1 and D_2. Let $\mathbf{N}$ be one of the normal vectors of the hyperplane $T_P\Sigma_1 = T_P\Sigma_2$. Let us denote by L_i the Weingarten map of the hypersurface Σ_i with respect to the normal vector $\mathbf{N}$ at the point P for $i = 1, 2$. Consider the one-parameter family of diffeomorphisms generated by a vector field X, which transform D_1 to D_1^t. Suppose that the eigenvalues of the symmetric operator $\pm(L_1 - L_2)$ are positive, where $\pm$ is the sign of the number $-\langle X(P), \mathbf{N}\rangle$. Then for small values of t, we have*

$$\mu(D_1^t \cap D_2) = \frac{\omega_{n-2}}{n^2-1} \frac{|\langle X(P), \mathbf{N}\rangle|^{\frac{n+1}{2}}}{\sqrt{|\det(L_1 - L_2)|}} (2t)^{\frac{n+1}{2}} + O\left(t^{\frac{n+2}{2}}\right), \tag{1}$$

where μ is the volume measure of the Riemannian manifold, and ω_{n-2} denotes the $(n-2)$-dimensional measure of the $(n-2)$-dimensional Euclidean unit sphere.

We want to use Theorem 2.1 to compute the volume of slightly intersecting geodesic balls. For this purpose, we have to compute the Weingarten map of geodesic spheres, which appears in the theorem. There are several papers on the computation of the Weingarten map of geodesic spheres, see for example the papers [17] and [18].

First we express the Weingarten map of a small geodesic sphere with the help of Jacobi fields. Let $\gamma\colon (a,b) \to M$ be a unit-speed geodesic curve, and suppose that $0 \in (a,b)$. For $0 \neq r \in (a,b)$, denote by $\Sigma_\gamma(r)$ the geodesic sphere of radius $|r|$ centered at $\gamma(r)$. If $|r|$ is sufficiently small, then $\Sigma_\gamma(r)$ is a smooth hypersurface in

M passing through $\gamma(0)$. In such a case, denote by $L_\gamma(r)$ the Weingarten map of the hypersurface $\Sigma_\gamma(r)$ at the point $\gamma(0)$ with respect to the normal vector $\gamma'(0)$.

Proposition 2.2. *Let $0 \neq r \in (a,b)$ be a fixed number. Suppose that the restriction of the exponential map at $\gamma(r)$ to the closed ball of radius $|r|$ centered at the origin of $T_{\gamma(r)}M$ is a diffeomorphism onto its image. Let $\mathbf{v}$ be a tangent vector to $\Sigma_\gamma(r)$ at the point $\gamma(0)$. Let J be the Jacobi field along γ such that $J(r) = \mathbf{0}$ and $J(0) = \mathbf{v}$. Then $L_\gamma(r)(\mathbf{v}) = -J'(0)$.*

Let $\mathbf{e}_1, \dots, \mathbf{e}_n = \gamma'(0)$ be an orthonormal basis at the point $\gamma(0)$. Denote by $\hat{L}_\gamma(r)$ the matrix of the operator $L_\gamma(r)$ with respect to the basis $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$. Using Proposition 2.2, we see that $\hat{L}_\gamma$ has a simple pole at the origin and we can compute all the members of its Laurent series. Its Laurent series around the origin starts as follows:

$$\hat{L}_\gamma(r) = I\frac{1}{r} + \frac{\hat{R}(0)}{3} r + O(r^2), \tag{2}$$

where $\hat{R}(0)$ denotes the matrix of the restriction of the Jacobi operator for the tangent vector $\gamma'(0)$ to the subspace $\gamma'(0)^\perp$ with respect to the basis $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$.

We can use the asymptotical formula (1) for the volume of two slightly intersecting balls to characterize Riemannian manifolds having the KP_2 or $KP_2^=$ property.

Let $\gamma\colon (a,b) \to M$ be a unit-speed geodesic and suppose that $0 \in (a,b)$. The operator $L_\gamma(r_1) - L_\gamma(r_2)$ is defined for small non-zero values of r_1 and r_2, and so is its determinant

$$D_\gamma(r_1, r_2) = \det(L_\gamma(r_1) - L_\gamma(r_2)).$$

Proposition 2.3 ([9])**.**

(i) *If M is a Riemannian manifold having the KP_2 property, then the germ of the function $D_\gamma(r_1, r_2)$ at the origin $(0,0) \in \mathbb{R}^2$ does not depend on γ.*

(ii) *If M is a Riemannian manifold having the $KP_2^=$ property, then the germ of the function $r \mapsto D_\gamma(r, -r)$ at $0 \in \mathbb{R}$ does not depend on γ.*

From this proposition, we can get the following theorem easily with the help of the formula (2).

Theorem 2.4 ([9])**.** *A complete Riemannian manifold having the KP_2 property is harmonic.*

If we assume only the $KP_2^=$ property, then we get the following theorem.

Theorem 2.5 ([9])**.** *A Riemannian manifold having the $KP_2^=$ property is Einstein.*

Though the results below imply the following theorem (as every symmetric harmonic space is 2-point homogeneous), it is interesting that we can get this claim directly.

Theorem 2.6 ([9]). *If a complete, simply connected symmetric Riemannian manifold has the $KP_2^=$ property, then it is 2-point homogeneous.*

Our next goal was to sharpen Theorem 2.4 by replacing the KP_2 property with the $KP_2^=$ property. This stronger result is found in the paper [10].

For this purpose, we use equivalent characterizations of D'Atri spaces. We say that a Riemannian manifold (M, g) is a *D'Atri space*, if the local geodesic symmetries are volume-preserving. For the points P and Q of a Riemannian manifold, denote by $h_P(Q)$ the mean curvature at Q of the sphere centered at P passing through Q. With this notation, we can formulate the following theorem about D'Atri spaces.

Theorem 2.7 ([1, 16]). *For an analytic Riemannian manifold (M, g), the following statements are equivalent:*

1. *M is a D'Atri space;*
2. *$h_P(Q) = h_Q(P)$ for all points $P, Q \in M$ sufficiently close;*
3. *$h_P(\exp_P(\mathbf{v})) = h_P(\exp_P(-\mathbf{v}))$ for each point $P \in M$ and each tangent vector $\mathbf{v} \in T_PM$ with sufficiently small norm;*
4. *$h_{\exp_P(\mathbf{v})}(P) = h_{\exp_P(-\mathbf{v})}(P)$ for each $P \in M$ and each tangent vector $\mathbf{v} \in T_PM$ with sufficiently small norm.*

First we prove that a Riemannian manifold with the $KP_2^=$ property is a D'Atri space.

Theorem 2.8 ([10]). *A Riemannian manifold having the $KP_2^=$ property is a D'Atri space.*

By Theorem 2.5, every Riemannian manifold with the $KP_2^=$ property is Einstein, thus, it is analytic by the Kazdan–DeTurck theorem, so we can apply Theorem 2.7 to it. With some further computation making use of analyticity in an essential way, we get the desired theorem:

Theorem 2.9 ([10]). *Every Riemannian manifold having the $KP_2^=$ property is harmonic.*

Z. I. Szabó proved in [20] that every connected, simply connected and complete harmonic manifold have the $KP_2^=$ property, so we can also formulate the following theorem.

Theorem 2.10. *A connected, simply connected and complete Riemannian manifold have the $KP_2^=$ property if and only if it is harmonic.*

These results are local results, we applied the KP_2 property for small balls. The following theorem gives a global result, but only in the case of constant curvature. The manifolds of constant curvature having the $KP_2^=$ property are almost simply connected.

Theorem 2.11. *Suppose that (M,g) is a connected, simply connected and complete Riemannian manifold of constant curvature, which have the $KP_2^=$ property. Then M is one of the spaces $\mathbb{H}^n_\kappa, \mathbb{E}^n, \mathbb{S}^n_\kappa, \mathbb{RP}^n_\kappa$.*

Though this theorem is formulated for the KP_2 property in the paper [12], the proof given there uses only the $KP_2^=$ property.

3 The intersection of three balls

Balázs Csikós and Dávid Kunszenti-Kovács showed in [12] that if a connected complete Riemannian manifold has the KP_3 property, then it is one of the simply connected spaces of constant curvature. Our goal is to extend this result to the $KP_3^=$ property. In this chapter we summarize the results of the paper [11].

We will use a weaker property instead of the $KP_3^=$ property, which says that the minimum covering radius of a triangle can depend only on the lengths of the sides. To check this property, we do not need the volume of the intersection of three balls, we need only to know whether the intersection is empty or not.

By a triangle in a manifold, we mean an *arbitrary* triple of points in the manifold. We define the minimum covering radius r_{ABC} of a triangle ABC in a connected Riemannian manifold M as the infimum of the radii r such that the intersection of the geodesic balls centered at A, B, and C with radius r is nonempty:

$$r_{ABC} = \inf \{ r \mid \mathcal{B}(A,r) \cap \mathcal{B}(B,r) \cap \mathcal{B}(C,r) \neq \emptyset \}.$$

If M is complete or one of the ball closures $\overline{\mathcal{B}}(A,\rho)$, $\overline{\mathcal{B}}(B,\rho)$, $\overline{\mathcal{B}}(C,\rho)$ is compact for a certain $\rho > r_{ABC}$, then we also have

$$\begin{aligned} r_{ABC} &= \min \left\{ r \;\middle|\; \overline{\mathcal{B}}(A,r) \cap \overline{\mathcal{B}}(B,r) \cap \overline{\mathcal{B}}(C,r) \neq \emptyset \right\} \\ &= \min \left\{ r \;\middle|\; \exists P \in M, \text{ such that } A,B,C \in \overline{\mathcal{B}}(P,r) \right\}. \end{aligned}$$

In a $KP_3^=$ manifold, the volume of the intersection $\mathcal{B}(A,r) \cap \mathcal{B}(B,r) \cap \mathcal{B}(C,r)$, consequently its emptiness or non-emptiness depends only on the lengths of the sides of the triangle, i.e., the geodesic distances between the vertices. Thus, the following proposition is obvious.

Proposition 3.1. *In a $KP_3^=$ manifold, the minimum covering radius of a triangle depends only on the lengths of the sides of the triangle.*

For a given connected Riemannian manifold M, choose a continuous function $\mathfrak{r}\colon M \to (0,\infty)$ such that for every $P \in M$, $\exp_P$ is defined on the ball of radius $\mathfrak{r}(P)$ centered at the origin of T_PM, and the ball $\mathcal{B}(P,r)$ is a geodesically convex set for any $r \leq \mathfrak{r}(P)$. We shall call a triangle *small* if it is contained in a ball $\mathcal{B}(P,\mathfrak{r}(P))$ for some $P \in M$.

Let us consider a 2-plane $\sigma < T_PM$, and a tangent vector $\mathbf{v} \in \sigma$ of length $r = \|\mathbf{v}\| < \mathfrak{r}(P)$, see Figure 1. Set $A = \exp_P(\mathbf{v})$ and $B = \exp_P(-\mathbf{v})$. Let $\mathbf{w} \in \sigma$

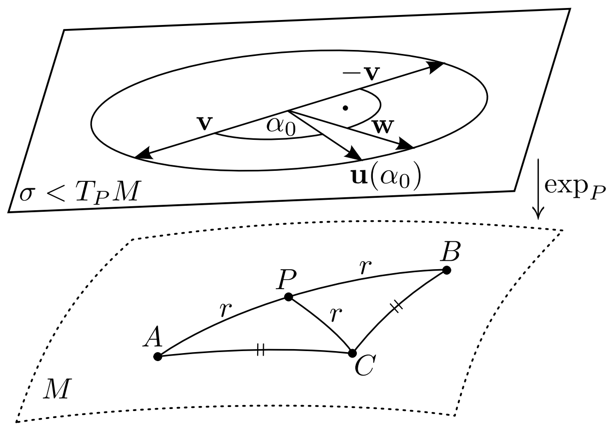

Figure 1

be one of the two tangent vectors of length r perpendicular to $\mathbf{v}$. Define the map $\mathbf{u}\colon [0,\pi] \to T_P M$ by $\mathbf{u}(\alpha) = \mathbf{v}\cos(\alpha) + \mathbf{w}\sin(\alpha)$. Obviously, we have $\mathbf{u}(0) = \mathbf{v}$ and $\mathbf{u}(\pi) = -\mathbf{v}$. The distance between A and $\exp_P(\mathbf{u}(\alpha))$ changes from 0 to $2r$ continuously and the distance between B and $\exp_P(\mathbf{u}(\alpha))$ changes from $2r$ to 0 continuously, so there is a value α_0 where these distances are equal. Set $C = \exp_P(\mathbf{u}(\alpha_0))$. Then $d(A,C) = d(B,C)$ by the construction. As $\|\mathbf{u}(\alpha)\| = r$ for all $\alpha \in [0,\pi]$, we also have $d(P,C) = r$. Thus, the minimum covering radius of the triangle ABC is r.

The points A, B and C depend on the choices of $P, \sigma, r, \mathbf{v}, \mathbf{w}, \alpha_0$. We shall call the ordered triple ABC the triangle constructed from the data $(P, \sigma, r, \mathbf{v}, \mathbf{w}, \alpha_0)$. For a given point $P \in M$, a 2-plane $\sigma < T_P M$ and a positive number $r < \mathfrak{r}(P)$, denote by $\Delta(P, \sigma, r)$ the set of all triangles that can be constructed from the data $(P, \sigma, r, \mathbf{v}, \mathbf{w}, \alpha_0)$ for some suitable tangent vectors $\mathbf{v}, \mathbf{w}$ and angle α_0.

Proposition 3.2. *If M has the property that the minimum covering radius of small triangles depends only on the lengths of the sides, then there is a function $\hat{a}\colon [0,\infty) \to \mathbb{R}$ such that $d(A,C) = \hat{a}(r)$ for any triangle ABC that can be constructed from some data $(P, \sigma, r, \mathbf{v}, \mathbf{w}, \alpha_0)$ with $r < \frac{\mathfrak{r}(P)}{3}$.*

The other tool for proving that $KP_3^=$ spaces have constant sectional curvature is Rauch's comparison theorem. We need a somewhat stronger version than the known version.

Theorem 3.3. *In the Riemannian manifolds M_1 and M_2, consider the unit-speed geodesics $\gamma_i\colon [0,l] \to M_i$ for $i = 1,2$. Assume that for no $t \in (0,l]$ is $\gamma_2(t)$ conjugate to $\gamma_2(0)$ along γ_2. Let J_i be Jacobi fields along γ_i for $i = 1,2$, such that $J_i(0)$ is tangent to γ_i and*

$$\langle \gamma_1'(0), J_1(0)\rangle = \langle \gamma_2'(0), J_2(0)\rangle,$$
$$\langle \gamma_1'(0), J_1'(0)\rangle = \langle \gamma_2'(0), J_2'(0)\rangle,$$
$$\|J_1'(0)\| = \|J_2'(0)\|.$$

Suppose that for all $t \in (0,l]$ and arbitrary tangent vector $\mathbf{v}_2 \in T_{\gamma_2(t)}M_2$,

$$K_1(\mathrm{span}\{J_1(t), \gamma_1'(t)\}) \le K_2(\mathrm{span}\{\mathbf{v}_2, \gamma_2'(t)\}),$$

where K_i denotes the sectional curvature function of M_i for $i = 1,2$. Then for all $t \in [0,l]$, we have $\|J_1(t)\| \ge \|J_2(t)\|$.

The curvature condition is usually assumed for all planes generated by the tangent vectors $\mathbf{v}_1 \in T_{\gamma_1(t)}M_1$ and $\gamma_1'(t)$ in M_1, but it is not needed for all such planes. The proof given in the book [4] works for Theorem 3.3 as well.

We can modify the corollary with the help of the above theorem.

Theorem 3.4. *Let M_1, M_2 be Riemannian manifolds such that $\dim M_1 \le \dim M_2$ and consider a point $P_i \in M_i$ for $i = 1,2$, see Figure 2. Let $A\colon T_{P_1}M_1 \to T_{P_2}M_2$ be a*

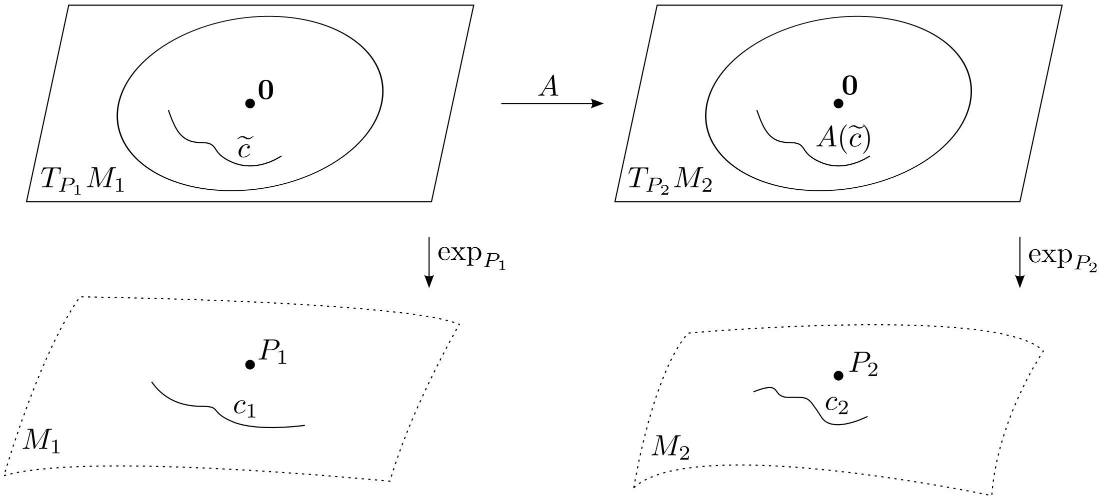

Figure 2

linear map preserving inner products. Choose r as small as $\exp_{P_1}|_{\mathcal{B}(\mathbf{0},r)}$ is an embedding and $\exp_{P_2}|_{\mathcal{B}(\mathbf{0},r)}$ is nonsingular. Consider a curve $c_1\colon [a,b] \to \exp_{P_1}(\mathcal{B}(\mathbf{0},r))$, and the unique curve $\widetilde{c}\colon [a,b] \to \mathcal{B}(\mathbf{0},r)$ such that $\exp_{P_1}\widetilde{c}(t) = c_1(t)$. We define the curve $c_2\colon [a,b] \to \exp_{P_2}(\mathcal{B}(\mathbf{0},r))$ by the formula $c_2(t) = \exp_{P_2}(A(\widetilde{c}(t)))$. Let $\Lambda_1\colon [0,1] \times [a,b] \to M$ be the singular rectangle given by $\Lambda_1(s,t) = \exp_{P_1}(s\widetilde{c}(t))$. Finally, suppose that $K_1(\sigma_1) \le K_2(\sigma_2)$ is satisfied for all 2-planes $\sigma_2 < TM_2$ and all 2-planes $\sigma_1 \in \mathrm{S}_{P_1}(c_1)$, where

$$\mathrm{S}_{P_1}(c_1) = \left\{ \mathrm{im} T_{(s,t)}\Lambda_1 \;\middle|\; s \in [0,1], t \in [a,b] \textit{ and } \dim \mathrm{im} T_{(s,t)}\Lambda_1 = 2 \right\}.$$

Then we have $\ell(c_1) \ge \ell(c_2)$.

For the application of Rauch's theorem, we need the following lemma about the set $\mathrm{S}_P(c)$ appearing in Theorem 3.4.

Lemma 3.5. *Denote by* $\mathrm{Gr}_2(TM)$ *the manifold of tangential 2-planes of* M*. For any open neighborhood* $U \subseteq \mathrm{Gr}_2(TM)$ *of a 2-plane* $\sigma < T_P M$*, there exists a positive number* ρ *such that for any* $r < \rho$ *and for any triangle* ABC *in* $\Delta(P, \sigma, r)$*, we have* $\mathrm{S}_P(\gamma_{AC}) \subseteq U$*, where* γ_{AC} *denotes the minimal geodesic joining* A *and* C*.*

Using Proposition 3.2 and Theorem 3.4, we can prove the following theorem.

Theorem 3.6 ([11])**.** *If* M *has the property that the minimum covering radius of small triangles depends only on the lengths of the sides, then* M *has constant sectional curvature. In particular, every* $KP_3^=$ *manifold is of constant sectional curvature.*

By the global Theorem 2.11, we can get a global theorem from Theorem 3.6.

Theorem 3.7 ([11])**.** *Suppose that* M *is a complete connected Riemannian manifold and* M *has the* $KP_3^=$ *property. Then* M *is a simply connected space of constant curvature.*

We need only to exclude the elliptic space. Though there is a proof in [13], we give a simpler one.

Remark 3.8. In a manifold of constant curvature κ, we can compute the function $\hat{a}(r)$ that appeared in Proposition 3.2. In this case, the angles APC and CPB are equal by symmetry, so they are equal to $\frac{\pi}{2}$. Then by the law of cosines,
in case $\kappa = 0$, $\hat{a}(r) = \sqrt{2}r$;
in case $\kappa < 0$, (when $\hat{a}(r) > \sqrt{2}r$) $\cosh\left(\hat{a}(r)\sqrt{-\kappa}\right) = \cosh\left(r\sqrt{-\kappa}\right)^2$;
in case $\kappa > 0$, (when $\hat{a}(r) < \sqrt{2}r$) $\cos\left(\hat{a}(r)\sqrt{\kappa}\right) = \cos\left(r\sqrt{\kappa}\right)^2$.
Though we cannot compute the value of κ from these equations explicitly, we know that one value of the function $\hat{a}$ determines the curvature of the manifold.

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