Mesh independent superlinear convergence of some iterative methods for elliptic problems

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Theses of the dissertation

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Introduction

When considering numerical treatment of elliptic partial differential equations, as approximating an unbounded problem, we are led to ill-behaved equations. In linear case and taking linearizations of nonlinear equaitons the occuring matrices are ill-conditioned. When using finite element discretization the refinement of the finite element subspace results in the increase of these matrices. To ease this problem an efficient tool is using preconditioning. With this method we essentially transform the equation to a subproblem which has better properties. In this work the preconditioning matrix is the discretization of the principle part of the equation, namely the discrete Laplacian, similarly to the continuous case where we use the principal part as a preconditioner and we are lead to a bounded problem on a suitable Sobolev space, since the occuring Sobolev space coincide with the enery space corresponding to the Laplacian.

We show that conjugate gradient (CG) method applied to the classes of symmetric linear elliptic mentioned in the dissertation bear superlinear convergence, For nonsymmetric equations the conjugate gradient method applied to the normal equations (CGN) bear similar superlinear convergence. For nonlinear equations we show that the a well-chosen quasi-Newton method has superlinear convergence and the CG solution of their linear subproblems has also superlinear convergence.

Preliminary and short summary

The first superlinear convergence result of the CG is due to Hayes [10]. Since then many similar results have been published for the CG method and to its generalizations, in the dissertation we use the results in [5, 19].

Relating th this subject the analysis of preconditioning of elliptic partial differential operators on an operator level has begun in the 1980s. For classes of elliptic boundary problems the the classification of the preconditioning operators that can be used in order to achieve superlinear convergence of the CG [9, 8]. Later came results that covered superlinear convergence of the CG applied to the finite element discretizations, see in e.g. [6, 7]. The proofs are based on the close connection between the finite element solution and the weak solutions of the original continuous equaition, thus the efficient tools of functional analysis and the theory of Sobolev spaces can be applied.

First we consider linear and nonlinear symmetric elliptic systems. In both cases the superlinear convergence of the proposed methods are proved giving explicit order of convergence. In these cases the mesh independent convergence estimate is proved using the Hilbert-Schmidt norm of the inverse of the Laplacian.

Next we consider nonsysmmetric elliptic and interface problems, both nonlinear. We prove superlinear convergence of the solution of the linear subproblem of a quasi-Newton method using a detalied analysis of the occuring operators. And finally we give a short summary of the proved orders of convergence.

Short summary of the applied methods

The equations mentioned in the dissertation are of the form:

$$Su + Q(u) = f,$$

where the operators S and Q act on a suitable H Hilbert space, S is an elliptic operator and Q contains the smaller order terms or the boundary conditions. The weak form is:

$$\langle u + S^{-1}Q(u), v\rangle_S = \langle f, v\rangle,$$

where $S^{-1}Q$ is defined on the enery space H_S. In the linear case or when considering linearization of a nonlinear problem the operator Q is linear, and positive (because of our assumtions on the equations).
The finite element discretization is as follows: Let $V_k = \text{span}\{\varphi_1, \ldots, \varphi_k\} \subset H_S$ be a finite dimensional subspace, then the finite element solutions is of the form $u_h = \sum_{i=1}^k c_i\varphi_i$ which satisfies

$$\langle Su_h + Q(u_h), \varphi_i\rangle = \langle f, \varphi_i\rangle, \quad i = 1, \ldots, k,$$

After preconditioning with the matrix $S_k = \{\langle\varphi_i, \varphi_j\rangle_S\}_{i,j=1}^k$ We have the following algebraic equation:

$$\mathbf{c} + N(\mathbf{c}) = \mathbf{f}. \tag{1}$$

In linear case using the notations

$$Q_h = \{\langle Q\varphi_i, \varphi_j\rangle\}_{i,j=1}^k \text{ and } f_h = \{\langle f, \varphi_i\rangle\}_{i=1}^k$$

we have:

$$(I_h + S_h^{-1}Q_h)\mathbf{c} = S_h^{-1}f_h. \tag{2}$$

We solve this equation with the CG method in the using the S_h inner product. Q. This equation can be seen as a perturbation of the identity, and for these types of equations the CG has favourable convergence properties.

Theorem 1 ([5]). *Let e_n denote the error of the nth approximtion. Then we have*

$$\frac{\|e_n\|_{S_h+Q_h}}{\|e_0\|_{S_h+Q_h}} \leq \left(\frac{3}{2n} \left|\left|\left| S_h^{-1} Q_h \right|\right|\right|^2 \right)^{n/2},$$

where $|||\cdot|||$ denotes the Frobenius norm of a matrix.

Theorem 2 ([19]). *Let r_n denote the nth residual. Then we have*

$$\frac{\|r_n\|}{\|r_0\|} \leq \left(\frac{2}{n} \sum_{i=1}^{n} \left| \lambda_i(S_h^{-1} Q_h) \right| \right)^n.$$

In the dissertation we show that with our assumptions the expressions in the brackets of the right hand side tend to zero.

For the solution of nonlinear equations we use the following quasi-Newton method:

Theorem 3 ([11]). *Let the function $F : X \to X$ be differentiable, and have the properties*

(i) $\|F'(u)h\| \geq \lambda \|h\| \quad (u, b \in X)$ *with some constant* $\lambda > 0$ *independent of* u, h,

(ii) $\|F'(u) - F'(v)\| \leq L\|u - v\| \quad (u, v \in X)$ *with some constant* $L > 0$ *independent of* u, v.

(ii') $\|F'(u) - F'(v)\| \leq L(r)\|u - v\| \quad (u, v \in X, \|u\| < r, \|v\| < r)$ *holds for some* $L : \mathbb{R}^+ \to \mathbb{R}^+$ *nondecreasing function.*

Then the solution of $F(u) = b$ is obtained with the following procedure:

Require: $u_0 \in X$, $r_0 = b - F(u_0)$

while $\|r_n\| > \varepsilon$ ***do***

$r_n = b - f(u_n)$,

find p_n *such that*

$$\|F'(u_n)p_n - r_n\| \leq \delta_n \|r_n\| \text{ with } 0 < \delta_n \leq \delta_0 < 1$$

define $\tau_n = \min \left\{ 1, \frac{1-\delta_n}{(1+\delta_n)^2} \frac{\lambda^2}{L\|r_n\|} \right\}$,

update $u_{n+1} = u_n + \tau_n p_n$

end while

It has the following convergence properties:

- *The sequence (u_n) converges to the exact solution u^* as*

$$\|u_n - u^*\| \leq \|F(u_n) - b\| \to 0 \text{ monotonically.}$$

- *if $\delta_n \equiv \delta_0$ then we have linear convergence*
- *if $\delta_n \leq const \cdot \|F(u_n) - b\|^\gamma$ $(0 < \gamma \leq 1)$ then the convergence is locally of order $1+\gamma$, that is the convergence is linear for n_0 steps, until $\|F(u_n) - b\| \leq \varepsilon$, where ε is at most $(1-\delta_n)\frac{1}{2L}$, and further on $\|u_n - u^*\| \leq Cq^{(1+\gamma)^{n-n_0}}$ holds*
- *when only (ii') holds, then choosing a sufficiently large R for the initial guess u_0 we have similar convergence estimates.*

We show that for the classes of nonlinear equations that we consider the conditions of this theorem hold, independent of the finite element subspace.

Summary of results

Symmetric semilinear systems of equations

In the first chapter we consider the following class of semilinear systems of the form:

$$\begin{cases} -\Delta \underline{u}(x) + f(x, \underline{u}(x)) = \underline{g}(x) \\ \underline{u}\big|_{\partial\Omega} = \underline{0}, \end{cases}$$

where $\underline{u} = (u_1, u_2, \ldots, u_s)$, $\underline{g} = (g_1, g_2, \ldots, g_s)$, with assumptions:

[P1] $\partial\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is piecewise C^2 and Ω is locally convex at the corners,

[P2] $g_i \in L^2(\Omega)$ $(i = 1, 2, \ldots, s)$ on Ω,

[P3] $f : \Omega \times \mathbb{R}^s \to \mathbb{R}^s$, for a.e. $x \in \Omega$ $f(x,\xi)$ has a potential $\psi : \Omega \times \mathbb{R}^s \to \mathbb{R}$, i.e. $f = \partial_\xi \psi$ and is differentiable w.r.t. ξ, and in these points the Jacobians are symmetric positive semidefinite,

[P4] for a.e. $x \in \Omega$ the Jacobians $\partial_\xi f(x,\xi)$ are uniformly bounded in ξ by a symmetric matrix $M(x)$, where the eigenvalues $\mu_j(x)$ of $M(x)$ are bounded $0 \leq \mu_j(x) \leq c_1$, with some constant $c_1 > 0$,

[P4'] the eigenvalues $\lambda_j^{(f)}(x,\xi)$ $(j = 1, \ldots, s)$ of the Jacobians $\partial_{\underline{\xi}} f(x,\xi)$ are bounded as follows

$$0 \leq \lambda_j^{(f)}(x,\xi) \leq c_2 + c_3 \sum_{j=1}^{s} |\xi|^{p-2},$$

for some constants $c_2, c_3 > 0$ and $p \geq 2$,

[P5] the derivative of f is Lipschitz continuous, that is there exists a constant C that $\|\partial_\xi f(x,\xi_1) - \partial_\xi f(x,\xi_2)\|_2 \leq C\|\xi_1 - \xi_2\|_2$ for a.e. $x \in \Omega$, [P5'] the derivative of f is locally Lipshcitz continuous, that is there exists a function $C : (0,\infty) \to (0,\infty)$ that $\|\partial_\xi f(x,\xi_1) - \partial_\xi f(x,\xi_2)\|_2 \le C(r)\|\xi_1 - \xi_2\|_2$ for a.e. $x \in \Omega$ if $\|\xi_1\|, \|\xi_2\| \le r$.

With these assumptions the quasi-Newton method for the respective finite element equation (1) converges, and the expressions L and $L(r)$ describing the Lipschitz property can be chosen independently of the finite element subspace. The linear subproblem of the quasi-Newton method has the weak form

$$\langle \underline{w} + N'(\underline{u})\underline{w}, \underline{v}\rangle_S = \int_\Omega \nabla \underline{w}(x) \cdot \nabla \underline{v}(x) + \partial_\xi f(x, \underline{u}(x))\underline{w}(x)\underline{v}(x)\, \mathrm{d}x.$$

So the preconditioned equation is the pertturbation of the identity, hence the following convergence estimate can be obtained:

Theorem 4. *Let $\underline{r}_k$ denote kth residual. Then we have*

$$\frac{\|\underline{r}_k\|_{S_k}}{\|\underline{r}_0\|_{S_k}} \le \left(\frac{K\, |||C|||^2}{k}\right)^{k/2},$$

where K is a suitable constant and $C = (-\Delta^{-1}, \dots, -\Delta^{-1})$ is the s-tuple of the inverse Laplacians, and $|||C|||$ denotes the Hilbert-Schmidt norm of C. This latter is finite for dimensions $d = 1, 2, 3$, thus we have mesh independent superlinear convergence.

Nonlinear nonsymmetric and interface problems

In the second chapter of the dissertation we replace the convergence estimate using the Hilbert-Schmidt norm with the one in Theorem 2. So that we can prove superlinear convergence for a broader class of problems.

We consider nonlinear elliptic transport systems of the form

$$\left.\begin{aligned} &-\mathrm{div}\,(K_i\,\nabla u_i) + \mathbf{b_i}\cdot\nabla \mathbf{u_i} + \mathbf{f_i}(\mathbf{x}, \mathbf{u_1}, \dots, \mathbf{u_l}) = \mathbf{g_i} \\ &u_{i\,|\partial\Omega} = 0 \end{aligned}\right\} \qquad (i = 1, \dots, l)$$

on a bounded domain $\Omega \subset \mathbf{R}^d$ ($d = 2$ or 3) under the following assumptions:

(i) $K_i \in L^\infty(\Omega)$, $\mathbf{b_i} \in \mathbf{C^1}(\overline{\mathbf{\Omega}})^{\mathbf{d}}$ and $g_i \in L^2(\Omega)$ $(i = 1, \dots, l)$, further, the function $f = (f_1, \dots, f_l) : \Omega \times \mathbf{R}^l \to \mathbf{R}^l$ is measurable and bounded w.r. to the variable $x \in \Omega$ and C^1 in the variable $\xi \in \mathbf{R}^l$.

(ii) there is $m > 0$ such that $K_i \ge m$ holds for all $i = 1, \dots, l$, further, using the notation $f'_\xi(x,\xi) := \frac{\partial f(x,\xi)}{\partial \xi}$,

$$f'_\xi(x,\xi)\,\eta\cdot\eta - \frac{1}{2}\left(\max_i \operatorname{div}\mathbf{b_i(x)}\right)|\eta|^{\mathbf{2}} \geq \mathbf{0}$$

for any $(x,\xi)\in\Omega\times\mathbf{R}^l$ and $\eta\in\mathbf{R}^l$.

(iii) let $3\leq p$ (if $d=2$) or $3\leq p<6$ (if $d=3$), then there exist constants $c_1, c_2 \geq 0$ such that for any (x,ξ_1) and $(x,\xi_2)\in\Omega\times\mathbf{R}^l$,

$$\left\|f'_\xi(x,\xi_1)-f'_\xi(x,\xi_2)\right\| \leq \Big(c_1+c_2\,(\max\,|\xi_1|,|\xi_2|)^{p-3}\Big)|\xi_1-\xi_2|.$$

(iv) let p be defined as in (iii), then there exist constants $c_3, c_4>0$ such that for any (x,ξ_1) and $(x,\xi_2)\in\Omega\times\mathbf{R}^l$,

$$|f'_\xi(x,\xi)|\leq c_3+c_4|\xi|^{p-2}.$$

Similarly as before the quasi-Newton method converges superlinearly. As the linear subproblem is nonsymmetric, the CG method cannot be applied directly, but it can be applied to the normal equations leading to the CGN method. Analysing the eqigenvalues of the derivative of the nonlinear part and using the Gelfand numbers of Sobolev space embeddings [18] we proved the following:

Theorem 5. *Let r_k denote the kth residual. Then we have*

$$\left(\frac{\|r_k\|_{S_k}}{\|r_0\|_{S_k}}\right)^{1/k} = O(k^{-1/p}) \quad (\textit{if } d=2), \quad O(k^{(6-p)/(6p)}) \quad (\textit{if } d=3).$$

Finally we considered a class of interface problems. Interface problems arise in various branches of material science, biochemistry, multiphase flow etc. Such models often describe a situation when two distinct materials are involved with different conductivities or densities, another important example is from localized reaction-diffusion problems [12, 13]. Many special numerical methods have been designed for interface problems, e.g. those involving monotone iterations, see, e.g., [12, 16, 14, 15]. When one employs a fine mesh to obtain an accurate approximation, the arising large-scale system has a large condition number too, which in fact tends to infinity as the mesh parameter approaches zero.

0.0.1 Formulation of the problem

We consider nonlinear interface problems of the following type:

$$\begin{cases} -\mathrm{div}\ (A(x)\nabla u) + q(x,u) &= f(x) \quad \text{in } \Omega \setminus \Gamma, \\ [u]_\Gamma &= 0 \quad \text{on } \Gamma, \\ \left[A(x)\frac{\partial u}{\partial \nu}\right]_\Gamma + s(x,u) &= \gamma(x) \quad \text{on } \Gamma, \\ u &= g(x) \quad \text{on } \partial\Omega, \end{cases}$$

where $[u]_\Gamma$ and $\left[A(x)\frac{\partial u}{\partial \nu}\right]_\Gamma$ denote the jump (i.e. the difference of the limits from the two sides of the interface Γ) of u and $A(x)\frac{\partial u}{\partial \nu}$, respectively. In the case of autocatalytic reactions, the nonlinearities often have the form $q(x,\xi) = c_1 + c_2\xi^\alpha$ (and similarly for $s(x,\xi)$).

(A1) Ω is a bounded open domain in $\mathbb{R}^d$ ($d = 2$ or 3), the interface $\Gamma \subset \Omega$ and the boundary $\partial\Omega$ are piecewise smooth and Lipschitz continuous 1-codimensional surfaces.

(A2) $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$, for a.e. $x \in \Omega$ $A(x)$ is symmetric and it satisfies the usual condition of uniform ellipticity

$$0 < \mu_0|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \mu_1|\xi|^2$$

for some positive numbers μ_0, μ_1.

(A3) The scalar functions $q : \Omega \times \mathbb{R} \to \mathbb{R}$ and $s : \Gamma \times \mathbb{R} \to \mathbb{R}$ are measurable and bounded w.r.t. their first variable $x \in \Omega$ (resp. $x \in \Gamma$) and continuously differentiable w.r.t. their second variable $\xi \in \mathbb{R}$. Further, $f \in L^2(\Omega)$, $\gamma \in L^2(\Gamma)$ and $g \in H^1(\Omega)$.

(A4) Let $2 \le p_1$ if $d = 2$ or $2 \le p_1 < 6$ if $d = 3$, further, let $2 \le p_2$ if $d = 2$ or $2 \le p_2 < 4$ if $d = 3$. There exist constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \ge 0$ such that for any $x \in \Omega$ (or $x \in \Gamma$, resp.) and $\xi \in \mathbb{R}$

$$0 \le \partial_\xi q(x,\xi) \le \alpha_1 + \beta_1|\xi|^{p_1-2}, \qquad 0 \le \partial_\xi s(x,\xi) \le \alpha_2 + \beta_2|\xi|^{p_2-2}.$$

We showed that the quasi-Newton method has the same convergence properties as proved for the problems before. Further using the Gelfand numbers of embeddings of Sobolev spaces [18, 17] we have proved the following convergence estimate for the linear subproblem:

Theorem 6. *Let r_k denote the kth residual. Then we have*

$$\left(\frac{\|r_k\|_{S_k}}{\|r_0\|_{S_k}}\right)^{1/k} = O(k^{-1/d+(p-2)/2p} + k^{-1/(d-1)}).$$

List of publications forming the basis of the dissertation

[1] Antal, I. Mesh independent superlinear convergence of the conjugate gradient method for discretized elliptic systems, Hung. Electr. Jou. Sci. HU ISSN 1418-7108: HEJ Manuscript no.: ANM-080107-A

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