# Bridges and Profunctors

Bertalan Pécsi

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ELTE University, Budapest
PhD School of Mathematics
(school leader: *Miklós Laczkovich*)
Theoretical Mathematics Program
(program leader: *András Szűcs*)

Supervisor: *Ildikó Sain*

**Abstract**

The main aim of the dissertation is to present a way for defining bicategories, double categories, lax and colax functors between them, where the coherence pentagon and the lax comparison cells are *implicitly encoded* in the system. Besides this, we also study related structures such as Morita contexts. For bicategories we follow Tom Leinster's *unbiased* approach, and we use profunctors regarded as categories (by their *collage*) and reflections therein.

The main theses of the dissertation, section by section:

## 1 *Bicategories*

In def. 1.1. we give an elementary interpretation of Tom Leinster's '*unbiased bicategory*', [Leinster]. (We will mostly omit the prefix 'unbiased'.) According to this, an (unbiased) bicategory is given by its objects, arrows between objects, and 2-cells between arrows ( $\cdot \overset{\rightarrow}{\underset{\rightarrow}{\Downarrow}} \cdot$ ) which can be composed 'vertically' (for each pair of objects, the arrows between them and their 2-cells form a category). Also, the arrows can be composed 'horizontally', though this composition is only *weakly associative* (associative up to coherent isomorphisms). This horizontal composition of arrows is given by a weakly associative *family of operations*: an arrow is assigned to each path of the underlying graph, in a functorial way. Paths of length 0 are also considered, as objects. That way, the composition and the unit can be handled together. Moreover, the coherence axiom requires commuting *squares* only instead of pentagons.

Then some bicategorical notions are introduced, such as *adjoint* pair of arrows, internal *monoid*, its action, internal *bimodule*.

In example 1.1.7., among others, we present the bicategory $\mathbb{S}\mathrm{pan}$. Its objects are the sets, and an arrow from set $A$ to set $B$ is a span $A \leftarrow E \rightarrow B$ of functions, considered as a *bipartite graph*, $E$ being the set of edges. In this reading, a 2-cell of $\mathbb{S}\mathrm{pan}$ is a morphism of graphs which fixes all the vertices.

Example 1.1.14: Internal monoids in the bicategory $\mathbb{S}\mathrm{pan}$ are just the categories.

## 2 *Profunctors*

**Definition 2.1.** We call a category $\mathcal{H}$ a ***bridge*** between categories $\mathcal{A}$ and $\mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ (or their isomorphic copies) are disjoint full subcategories of $\mathcal{H}$, and $\mathrm{Ob}\mathcal{H} = \mathrm{Ob}\mathcal{A} \cup \mathrm{Ob}\mathcal{B}$. In notation: $\mathcal{H}\colon\mathcal{A} \rightleftharpoons \mathcal{B}$.
We call the arrows of $\mathcal{H} \setminus (\mathcal{A} \cup \mathcal{B})$ *heteromorphisms*.

A bridge $\mathcal{H}$ is called *directed* from $\mathcal{A}$ to $\mathcal{B}$ (in notation $\mathcal{H}\colon\mathcal{A} \twoheadrightarrow \mathcal{B}$), if all its heteromorphisms are of the form $A \rightarrow B$.

The directed bridges $\mathcal{A} \nrightarrow \mathcal{B}$ incorporate both the internal bimodules of $\mathbb{Span}$, and the functors of the form $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{S}et$ (so called *profunctors*), see thm. 2.3. In light of this, from now on, we will identify profunctos and directed bridges. The correspondent directed bridge is also called the '*collage*' of a profunctor.

Profunctors as arrows between categories, form a bicategory $\mathbb{Prof}$, its 2-cells are morphisms of bridges, i.e., functors being identity on both base categories. Next, we sketch the two canonical embeddings of the bicategory $\mathbb{Cat}$ of categories, functors and natural transformations into $\mathbb{Prof}$: every functor $F : \mathcal{A} \to \mathcal{B}$ determines a profunctor $F_* : \mathcal{A} \nrightarrow \mathcal{B}$ and a profunctor $F^* : \mathcal{B} \to \mathcal{A}$.

**Theorem 2.7.** Let $\mathcal{F} : \mathcal{A} \nrightarrow \mathcal{B}$ be a profunctor. Then the following statements hold:
*a)* There is a functor $F : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{F} \cong F_*$, if and only if $\mathcal{B}$ is a reflective subcategory of $\mathcal{F}$.
*b)* There is a functor $G : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F} \cong G^*$, if and only if $\mathcal{A}$ is a coreflective subcategory of $\mathcal{F}$.
*c)* If both condition holds then the functors $F$ and $G$ constructible by the universal properties will be adjoint: $F \dashv G$. In this case, $F_* \cong G^*$.

**Corollary 2.9.** Consider the category $\mathcal{A}dj$ with objects the categories, and with morphisms the adjoint pairs of functors $\underset{\mathcal{A} \to \mathcal{B} \to \mathcal{A}}{\langle F , G \rangle} :\ F \dashv G$. We define two of its subcategories:

$$\begin{aligned} \mathcal{C}orefl &:= \{\langle F, G\rangle \in \mathcal{A}dj \mid F \text{ is full embedding}\} \\ \mathcal{R}efl &:= \{\langle F, G\rangle \in \mathcal{A}dj \mid G \text{ is full embedding}\}. \end{aligned}$$

Then we have

$$\mathcal{A}dj = \mathcal{C}orefl \cdot \mathcal{R}efl,$$

in the sense that, $\forall\ F \dashv G$ adjunction $\exists\ F_1 \dashv G_1$ and $F_2 \dashv G_2$, such that $F = F_1 F_2,\ \ G = G_2 G_1,$ and $F_1$ and $G_2$ are full embeddings.

## *3 Equivalence bridges*

We can characterize equivalence and Morita equivalence of categories by means of special kinds of bridges:

**Theorem 3.4.** Categories $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* in $\mathbb{Cat}$ if and only if there exists a bridge $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ in which every $A$ is isomorphic to some $B$ and every $B$ is isomorphic to some $A$ within $\mathcal{H}$ (where $A \in \mathrm{Ob}\mathcal{A}$ and $B \in \mathrm{Ob}\mathcal{B}$).

**Theorem 3.9.** Categories $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* (i.e., are equivalent in $\mathbb{Prof}$) if and only if there exists a bridge $\mathcal{M} : \mathcal{A} \rightleftharpoons \mathcal{B}$ in which every arrow can be written as composition of heteromorphisms.

## 4 Morita contexts

We introduce El Kaoutit's generalization of *Morita contexts* of ring theory, [El Kaoutit], that also generalizes bridges, [Pecsi].

**Definition 4.1.** Let $f : A \to B$ and $g : B \to A$ be arrows in a bicategory, equipped with '*constructor*' 2-cells $\mu : fg \Longrightarrow 1_A$ and $\nu : gf \Longrightarrow 1_B$. We say that these form a ***Morita context***, provided

$$\mu f = f\nu \quad \text{and} \quad g\mu = \nu g.$$

Then we sketch El Kaoutit's theorem (thm. 4.3) which states that, starting from an arbitrary bicategory $\mathbb{B}$, its Morita contexts as arrows form another bicategory that we denote by $\mathbf{Mrt}\mathbb{B}$.

**Proposition 4.5.** If a constructor 2-cell of a Morita context is left invertible then it is isomorphism.

These Morita contexts straightly correspond to adjunctions with invertible unit. Moreover, those Morita contexts in which both constructor 2-cells are invertible, correspond to adjoint equivalences, and are called '*strict*'.

**Theorem 4.7.** The following conditions are equivalent for a Morita context $\mathbb{f}$:
*a)* $\mathbb{f}$ is strict.
*b)* There is a 2-cell in $\mathbf{Mrt}\mathbb{B}$ from a strict Morita context to $\mathbb{f}$. [Such a 2-cell then must be invertible.]
*c)* $\mathbb{f}$ is an equivalence arrow in $\mathbf{Mrt}\mathbb{B}$.

**Corollary 4.8.** Two objects are equivalent in $\mathbf{Mrt}\mathbb{B}$ if and only if they are equivalent in $\mathbb{B}$.

**Corollary 4.9.** The unit of any adjunction in $\mathbf{Mrt}\mathbb{B}$ is invertible (hence, it can also be viewed as a Morita context).

## 5 Double categories

If we extend a bicategory by a 'vertical structure' enabling 2-cells between not necessarily parallel arrows, then we could arrive to the notion of *pseudo double category*. These 2-cells will be briefly called *cells* to distinguish.
As we will mostly deal with weak *horizontal* composition, we will usually omit the prefix ' *pseudo*'.
Note that, in contrast, the weak direction in [Gran-Pare2] is the vertical.

First, we introduce the notion of *double profunctor* from a bicategory $\mathbb{A}$ to a bicategory $\mathbb{B}$ (in notation $\mathbb{F} : \mathbb{A} \nrightarrow \mathbb{B}$, def. 5.1) in which there can be so called 'vertical' heteromorphisms from objects $A \in \mathrm{Ob}\mathbb{A}$ to objects $B \in$

ObB and '(hetero-)cells' from arrows of $\mathbb{A}$ to arrows of $\mathbb{B}$, bounded by vertical heteromorphisms from left and right. Now the composition of heteromorphisms with the original arrows of $\mathbb{A}$ or $\mathbb{B}$ is not required, but the cells should be composed by 2-cells of $\mathbb{A}$ 'from above' and by 2-cells of $\mathbb{B}$ 'from below' and should also be horizontally composed with other cells:

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[r, bend left=60, "\alpha"] \arrow[d] \arrow[dr, phantom, "\varphi"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[r, bend right=60, "\beta"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "a_1"] \arrow[d] \arrow[dr, phantom, "\varphi"] & \cdot \arrow[r, "a_2"] \arrow[d] \arrow[dr, phantom, "\psi"] & \cdot \arrow[d] \\
\cdot \arrow[r, "b_1"'] & \cdot \arrow[r, "b_2"'] & \cdot
\end{tikzcd}
$$

$$a_i, \alpha \in \mathbb{A}, \quad b_i, \beta \in \mathbb{B}$$

Having these, we define a *(pseudo)* ***double category*** as an internal monoid in the bicategory $\mathbb{BicProf}$ of double profunctors between bicategories. The monoid operation is regarded as vertical composition, and the unit 'embeds' the base bicategory (the underlying object of the internal monoid).
The heteromorphisms of a double category are simply called 'vertical arrows'.

For example, there arises a double category (denoted by $\mathbb{SET}$) over $\mathbb{Span}$, in which the vertical arrows are functions between sets and the cells are graph morphisms along these.

Internal bimodules of $\mathbb{BicProf}$ are called *double profunctors* (between internal monoids, i.e., double categories), and they form the bicategory $\mathbb{DbProf}$.

Similarly to the fact that a profunctor can be viewed as a category (by considering its collage), a double profunctor between double categories can also be viewed as a double category. This observation is heavily used in the following.

Analogously to section 2., we chartacterize lax and colax functors between bi- or double categories by means of coreflections and reflections in double profunctors, respectively, as follows.

**Definition 5.9.** Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. We call a triple $\langle \mathbb{F}, (\mathtt{k}_A)_A, (\kappa_a)_a \rangle$ a ***reflective double profunctor***, if $\mathbb{F} : \mathbb{A} \not\downarrow \mathbb{B}$ is a double profunctor and for every $A \in \mathrm{Ob}\mathbb{A}$, there is assigned a vertical reflection arrow $\mathtt{k}_A$ of $A$ to the vertical category of points of $\mathbb{B}$, and for every horizontal arrow $\underset{A \to A_1}{a} \in \mathbb{A}$, there is assigned a reflection cell $\kappa_a :$ $\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "\mathtt{k}_A"'] & \cdot \arrow[d, "\mathtt{k}_{A_1}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ to the vertical category of horizontal arrows of $\mathbb{B}$.

Next we prove that reflective double profunctors $\mathbb{A} \not\downarrow \mathbb{B}$ and colax functors $\mathbb{A} \longrightarrow \mathbb{B}$ determine each other (theorems 5.11, 5.14, 5.17). The corresponding

double profunctor for a colax functor $K$ is denoted by $K_*$.
Dually, the coreflective double profunctors $\mathbb{A} \nleftarrow \mathbb{B}$ and lax functors $\mathbb{B} \longrightarrow \mathbb{A}$ also determine each other (thm. 5.13). The corresponding double profunctor for a lax functor $L$ is denoted by $L^*$.

In proposition 5.15.*c)* we prove a conjecture of Gabriella Böhm: the Morita contexts within the bimodules and internal monoids of the horizontal bicategory of a double category are the same as lax functors from a simple 2-category which has two isomorphic objects with a single isomorphism pair between them and only trivial (identity) 2-cells. This fits in the series that describes lax functors from very simple (2-)categories, like the case of internal monoids or bimodules, see proposition 5.15., cf. [Koslowski].

**Proposition 5.16.** Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. Then there is a one-to-one correspondence between lax functors $\mathbb{A}^{co} \times \mathbb{B} \to \mathbb{SET}$ and double profunctors $\mathbb{A} \nleftarrow \mathbb{B}$ as defined above.

### *6 Colax/lax adjunctions*

We use a concrete doubly weak double category to extend the two canonical embeddings $\mathbb{C}\mathrm{at} \hookrightarrow \mathbb{P}\mathrm{rof}$ as one for double categories.

We embed the (strictly associative) double category $\mathbb{D}\mathrm{bl}$ of psuedo double categories and their lax and colax functors as horizontal and vertical arrows, introduced in [Gran-Pare2], into the (doubly weak) double category $\mathbb{Q}(\mathbb{D}\mathrm{b}\mathbb{P}\mathrm{rof})$ of Ehresmann quintets associated to the bicategory $\mathbb{D}\mathrm{b}\mathbb{P}\mathrm{rof}$. (Theorem 6.5.)

This embedding is horizontally contravariant, so the *orthogonal adjunctions* (in the sense of [Gran-Pare2]) of $\mathbb{D}\mathrm{bl}$ correspond to *companion pairs* of $\mathbb{Q}(\mathbb{D}\mathrm{b}\mathbb{P}\mathrm{rof})$, which gives an alternative proof for the following theorem, that was also formulated in [Pare] and [Fio-Gam-Kock].

**Theorem 6.6.** A colax functor $K$ and a lax functor $L$ are orthogonal adjoint in the double category $\mathbb{D}\mathrm{bl}$ if and only if $K_* \cong L^*$ in the bicategory $\mathbb{D}\mathrm{b}\mathbb{P}\mathrm{rof}$.

### *Appendix*

In part **A** we compare the original definition of bicategories, according to [Benabou] to the present interpretation of Leinster's unbiased bicategories (definitions A.1. and 1.1). See also [Leinster].

In part **B** we introduce the 'doubly weak double categories', that we call 'Verity double categories'.– in which both horizontal and vertical compositions are only weakly associative – and the pseudofunctors between them, in definitions B.1. and B.3. See as 'double bicategory' in [Morton] or [Verity].

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