Summary of the Ph.D. dissertation

Discrete methods in geometric measure theory

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April 2011

1. Introduction

The thesis addresses problems from the field of geometric measure theory. It turns out that discrete methods can be used efficiently to solve these problems.

In Chapter 2 we investigate the following question proposed by Tamás Keleti. How large (in terms of Hausdorff dimension) can a compact set $A \subset \mathbb{R}^n$ be if it does not contain some given angle α, that is, it does not contain distinct points $P, Q, R \in A$ with $\angle PQR = \alpha$? Or equivalently, how large dimension guarantees that our set must contain α?

We also study an approximate version of this problem, where we only want our set to contain angles close to α rather than contain the exact angle α. This version turns out to be completely different from the original one, which is best illustrated by the case $\alpha = \pi/2$. If the dimension of our set is greater than 1, then it must contain angles arbitrarily close to $\pi/2$. However, if we want to make sure that it contains the exact angle $\pi/2$, then we need to assume that its dimension is greater than $n/2$.

Another interesting phenomenon is that different angles show different behaviour. In the approximate version the angles $\pi/3$, $\pi/2$ and $2\pi/3$ play special roles, while in the original version $\pi/2$ seems to behave differently than other angles.

The investigation of the above problems led us to the study of the so-called acute sets. A finite set $\mathcal{H}$ in $\mathbb{R}^n$ is called an acute set if any angle determined by three points of $\mathcal{H}$ is acute. Chapter 3 of the thesis studies the maximal cardinality $\alpha(n)$ of an n-dimensional acute set. The exact value of $\alpha(n)$ is known only for $n \leq 3$. For each $n \geq 4$ we improve on the best known lower bound for $\alpha(n)$. We present different approaches. On one hand, we give a probabilistic proof that $\alpha(n) > c \cdot 1.2^n$. (This improves a random construction given by Erdős and Füredi.) On the other hand, we give an *almost* exponential constructive example which outdoes the random construction in low dimension ($n \leq 250$). Both approaches use the small dimensional examples that we found partly by hand ($n = 4, 5$), partly by computer ($6 \leq n \leq 10$).

Finally, in Chapter 4 we show that the Koch curve is tube-null, that is, it can be covered by strips of arbitrarily small total width.

Chapter 2 is based on [H1] and [H2]. The latter is a joint paper with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. Chapter 3 and 4 are based on [H3] and [H4], respectively. (For the sake of completeness some constructions due to Máthé are also included in the thesis.)

2. How large dimension guarantees a given angle?

An easy consequence of Lebesgue's density theorem claims that for any Lebesgue measurable set $A \subset \mathbb{R}^n$ with positive Lebesgue measure it holds that a similar copy of any finite configuration of points can be found in A.

What can be said about infinite configurations? Erdős asked whether there is a sequence $x_n \to 0$ such that a similar copy of this sequence can be found in every measurable set $A \subset \mathbb{R}$ with $\lambda(A) > 0$. This question is usually referred to as Erdős similarity problem and still unsolved.

And what about finite configurations in null sets? The following problem was also posed by Erdős. How large (in terms of Hausdorff dimension) can a set $A \subset \mathbb{R}^2$ be if there is no equilateral triangle with all three vertices in A? Falconer answered this question by showing that there exists a compact set A on the plane with Hausdorff dimension 2 such that A does not contain three points that form an equilateral triangle. In fact, it was shown in [5, 8, 9] that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a compact set (in $\mathbb{R}$ or in $\mathbb{R}^2$) of full Hausdorff dimension, which does not contain a similar copy of the three points. It is open whether the analogous result holds in higher dimension.

It would be interesting to find *patterns*, which can be found in every full dimensional set. In this chapter we investigate such a pattern. We say that a set $A \subset \mathbb{R}^n$ contains the angle α if there exist distinct points $P, Q, R \in A$ such that $\angle PQR = \alpha$. Keleti posed the following question: how large can a set $A \subset \mathbb{R}^n$ be if it does not contain α? If there is no restriction on A, then for any given $\alpha \in [0, \pi]$ one can use transfinite recursion to construct a full dimensional set not containing α, see Theorem 2.14. The problem is more interesting, though, if we restrict ourselves to, for example, compact sets. What is the smallest s for which $\dim(A) > s$ implies that A must contain α provided that $A \subset \mathbb{R}^n$ is compact? (Or equivalently, what is the maximal Hausdorff dimension s of a compact set $A \subset \mathbb{R}^n$ with the property that A does not contain the angle α?) This minimal (maximal) value of s will be denoted by $C(n, \alpha)$. It is not hard to show that $C(n, \alpha) \leq n - 1$ for arbitrary α, in other words, if the Hausdorff dimension of a compact set $A \subset \mathbb{R}^n$ is greater than $n - 1$, then A contains every angle $\alpha \in [0, \pi]$.

As far as lower bounds are concerned, the line segment shows that $C(n, \alpha) \geq 1$ for any $\alpha \in (0, \pi)$. Our first goal is to improve on this obvious lower bound by constructing a compact set of Hausdorff dimension greater than 1 which does not contain some angle $\alpha \in (0, \pi)$.

Theorem 2.1. *There is a* $\delta_0 > 0$ *such that for any* $0 < \delta \leq \delta_0$ *there exists a self-similar set in* $\mathbb{R}^n$ *of dimension at least*

$$c_\delta n = c\delta^2 \log^{-1}(1/\delta) \cdot n$$

such that the angle determined by any three points of the set is in the δ*-neighbourhood of the set* $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

The above theorem readily implies that $C(n, \alpha) \geq c(\alpha)n$ given that $\alpha \in (0, \pi)$ and $\alpha \neq \pi/3, \pi/2, 2\pi/3$. The construction uses the following result due to Erdős and Füredi [4]. For any $\delta > 0$ there exist at least $(1 + c\delta^2)^n$ points in $\mathbb{R}^n$ such that the distance of any two is between 1 and $1 + \delta$. (This result is also related to the problems studied in Chapter 3.)

What about the exceptional angles $\pi/3$, $\pi/2$, $2\pi/3$? Our next goal is to prove that there exist self-similar sets in $\mathbb{R}^n$ with large dimension that contain neither $\pi/3$, nor $2\pi/3$. We start with constructing a discrete set of points. We need to find as many points P_i as possible such that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$. We were inspired by the following r-colouring of the complete graph on 2^r vertices. Let $C_1, \ldots, C_r$ denote the colours and let us associate to each vertex a 0-1 sequence of length r. Consider the edge between the vertices corresponding to the sequences $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$. We colour this edge with C_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k \neq j_k$. Let us denote this coloured graph by $\mathcal{G}_r$. This colouring has the property that there is no monochromatic triangle in the graph. Moreover, every triangle has two sides with the same colour and a third side with a different colour of higher index. (This is a folklore graph colouring showing that the multicolour Ramsey number $R_r(3)$ is greater than 2^r.)

The idea is to realize $\mathcal{G}_r$ geometrically in the following manner: the vertices of the graph will be represented by points of a Euclidean space and edges with the same colour will correspond to equal distances. The next lemma claims that $\mathcal{G}_r$ can be represented in the above sense.

Lemma 2.2. *Let* $l_1 \geq l_2 \geq \ldots \geq l_r > 0$ *be a decreasing sequence of positive reals. By* $\mathcal{I}_r$ *we denote the set of* 0-1 *sequences of length* r. *Then* 2^r *points* $P_{i_1,\ldots,i_r}$, $(i_1, \ldots, i_r) \in \mathcal{I}_r$ *can be given in some Euclidean space in such a way that for two distinct* 0-1 *sequences* $(i_1, \ldots, i_r) \neq (j_1, \ldots, j_r)$ *the distance of* $P_{i_1,\ldots,i_r}$ *and* $P_{j_1,\ldots,j_r}$ *is equal to* l_k *where* k *denotes the first index where the sequences differ, that is,* $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k \neq j_k$.

The proof of the next theorem uses the above lemma as well as the well-known Johnson-Lindenstrauss lemma [7].

Theorem 2.3. *There exist absolute constants* $c, C > 0$ *such that for any* $0 < \delta < \varepsilon < 1$ *with* $\varepsilon/\delta > C$ *there exists a self-similar set of dimension*

$$s \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}$$

in a Euclidean space of dimension

$$n \leq \frac{C\varepsilon}{\delta^3}$$

such that any angle determined by three points of the set is inside the ε*-neighbourhood of* $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ *but outside the* δ*-neighbourhood of* $\{\pi/3, 2\pi/3\}$.

By fixing a small ε and setting $\delta = c/\sqrt[3]{n}$ in the above theorem, we obtain the following corollaries. As we will see, the second one is surprisingly sharp.

Corollary 2.4. *A self-similar set* $K \subset \mathbb{R}^n$ *can be given such that the dimension of* K *is at least*

$$s \geq \frac{c\sqrt[3]{n}}{\log n},$$

and K *does not contain the angle* $\pi/3$ *and* $2\pi/3$ *(moreover,* K *does not contain any angle in the* $c/\sqrt[3]{n}$*-neighbourhood of* $\pi/3$ *and* $2\pi/3$*).*

Corollary 2.5. *For any* $0 < \delta < 1$ *there exists a self-similar set* K *of dimension at least* $\frac{c}{\delta}/\log(\frac{1}{\delta})$ *in some Euclidean space such that* K *does not contain any angle in* $(\pi/3 - \delta, \pi/3 + \delta) \cup (2\pi/3 - \delta, 2\pi/3 + \delta)$.

The rest of this chapter is joint work with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. According to the following results large dimensional sets always contain angles close to $\pi/3$, $\pi/2$ and $2\pi/3$.

Theorem 2.6. *Any set* A *in* $\mathbb{R}^n$ $(n \geq 2)$ *with Hausdorff dimension greater than 1 contains angles arbitrarily close to the right angle.*

Theorem 2.7. *There exists an absolute constant* C *such that whenever* $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ *for some set* $A \subset \mathbb{R}^n$ *and* $\delta > 0$ *the following holds:* A *contains three points that form a* δ*-almost regular triangle, that is, the ratio of the longest and shortest side is at most* $1+\delta$.

As an immediate consequence, we can find angles close to $\pi/3$.

Corollary 2.8. *Suppose that* $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ *for some set* $A \subset \mathbb{R}^n$ *and* $\delta > 0$*. Then* A *contains angles from the interval* $(\pi/3 - \delta, \pi/3]$ *and also from* $[\pi/3, \pi/3 + \delta)$.

Remark 2.9. The above theorem and even the corollary is essentially sharp, see Corollary 2.5.

We mention that the above results remain valid even under somewhat weaker conditions (when Hausdorff dimension is replaced with upper Minkowski dimension).

To sum up the results we introduce the following function $\widetilde{C}$ depending on an angle $\alpha \in [0, \pi]$ and a small positive δ.

$$\widetilde{C}(\alpha, \delta) \stackrel{\text{def}}{=} \sup\{\dim(A) : A \subset \mathbb{R}^n \text{ for } \textbf{some } n; A \text{ is analytic;} \\ A \text{ does not contain any angle from } (\alpha - \delta, \alpha + \delta)\}.$$

It is shown in the thesis that $\widetilde{C}$ satisfies the symmetry property

$$\widetilde{C}(\alpha, \delta) = \widetilde{C}(\pi - \alpha, \delta).$$

The above constructions and results give essentially all the values of $\widetilde{C}(\alpha, \delta)$, see Table 1.

TABLE 1. Smallest dimensions that guarantee angle in $(\alpha - \delta, \alpha + \delta)$

α	$\widetilde{C}(\alpha, \delta)$	
$0, \pi$	$= 0$	
$\pi/2$	$= 1$	
$\pi/3, 2\pi/3$	$\approx 1/\delta$	apart from a multiplicative error $C \cdot \log(1/\delta)$
other angles	$= \infty$	provided that δ is sufficiently small

Let us now return to estimates on $C(n, \alpha)$. First we give a precise definition.

Definition 2.10. If $n \geq 2$ is an integer and $\alpha \in [0, \pi]$, then let

$$C(n, \alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that} \\ \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}.$$

We mention that we get the same definition if we consider analytic sets instead of compact sets.

The next theorem says that if we have an analytic set in $\mathbb{R}^n$ of Hausdorff dimension greater than $n - 1$, then it must contain every angle $\alpha \in [0, \pi]$.

Theorem 2.11. *If $n \geq 2$ and $\alpha \in [0, \pi]$, then $C(n, \alpha) \leq n - 1$.*

We can prove a better upper bound for $C(n, \pi/2)$.

Theorem 2.12. *If n is even, then $C(n, \pi/2) \leq n/2$. If n is odd, then $C(n, \pi/2) \leq (n + 1)/2$.*

For the sake of completeness we mention a construction due to András Máthé. Both constructions that we have seen so far (the one for general angles and the one for $\pi/3$, $2\pi/3$) have the property that they avoid not only the given angle α but also a small neighbourhood of α. The following construction does not have this property: even though the constructed set contains angles arbitrarily close to $\pi/2$, it succeeds to avoid $\pi/2$. It is based on number theoretic methods.

Theorem 2.13 (Máthé, [10])**.** *There exists a compact set $K \subset \mathbb{R}^n$ such that* $\dim(K) = n/2$ *and K does not contain the angle $\pi/2$.*

It follows from Theorem 2.12 that this result is sharp given that n is even.

We gathered the best known bounds for $C(n, \alpha)$ in Table 2.

Table 2. Best known bounds for $C(n, \alpha)$

α	lower bound	upper bound
$0, \pi$	$n-1$	$n-1$
$\alpha \in (0,\pi)$; $\alpha \neq \pi/2$	cn	$n-1$
$\pi/2$	$n/2$	$\lceil n/2 \rceil$

Finally, the next theorem shows that if there was no restriction on A in Definition 2.10, then $C(n, \alpha)$ would be n for any α.

Theorem 2.14. *Let $n \geq 2$. For any $\alpha \in [0, \pi]$ there exists $H \subset \mathbb{R}^n$ such that H does not contain the angle α, and H has positive Lebesgue outer measure. In particular,* $\dim(H) = n$.

3. Acute sets in Euclidean spaces

Around 1950 Erdős conjectured that given more than 2^d points in $\mathbb{R}^d$ there must be three of them determining an obtuse angle. The vertices of the d-dimensional cube show that 2^d points exist such that the angle determined by any three of them is at most $\pi/2$.

In 1962 Danzer and Grünbaum proved this conjecture [3]. They posed the following question in the same paper: what is the maximal number of points in $\mathbb{R}^d$ such that all angles determined are acute (in other words, this time we want to exclude right angles as well as obtuse angles). A set of such points will be called an *acute set* or *acute d-set* in the sequel.

The exclusion of right angles seemed to decrease the maximal number of points dramatically: they could only give $2d - 1$ points, and they conjectured that this is the best possible. However, this was only proved for $d = 2, 3$ [6].

Then in 1983 Erdős and Füredi disproved the conjecture of Danzer and Grünbaum. They used the probabilistic method to show the existence of an acute d-set of cardinality exponential in d. Their idea was to choose random points from the vertex set of the d-dimensional unit cube, that is $\{0,1\}^d$. Actually they even proved the following result: for any fixed $\delta > 0$ there exist exponentially many points in $\mathbb{R}^d$ with the property that the angle determined by any three points is less than $\pi/3 + \delta$. We used this result in the previous chapter to construct large dimensional sets such that each angle contained by the sets is close to one of the angles 0, $\pi/3$, $\pi/2$, $2\pi/3$, π.

We denote the maximal size of acute sets in $\mathbb{R}^d$ and in $\{0,1\}^d$ by $\alpha(d)$ and $\kappa(d)$, respectively; clearly $\alpha(d) \geq \kappa(d)$. Our goal in this chapter is to give good bounds for $\alpha(d)$ and $\kappa(d)$. The random construction of Erdős and Füredi implied the following lower bound for $\kappa(d)$ (thus for $\alpha(d)$ as well)

$$\kappa(d) > \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d. \tag{1}$$

The best known lower bound both for $\alpha(d)$ and for $\kappa(d)$ (for large values of d) is due to Ackerman and Ben-Zwi from 2009 [1]. They improved (1) with a factor $\sqrt{d}$:

$$\alpha(d) \geq \kappa(d) > c\sqrt{d}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{2}$$

We modify the random construction of Erdős and Füredi to obtain the following theorem.

Theorem 3.1.

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d,$$

that is, there exist at least $c \cdot 1.2^d$ points in $\mathbb{R}^d$ such that any angle determined by three of these points is acute. (If d is divisible by 5, then c can be chosen to be $1/2$, for general d we need to use a somewhat smaller c.)

We present another approach where we recursively construct acute sets. These constructions outdo Theorem 3.1 up to dimension 250. We show that this constructive lower bound is *almost* exponential in the following sense. Given any positive integer r, for infinitely many values of d we have an acute d-set of cardinality at least

$$\exp(d/\underbrace{\log\log\cdots\log}_{r}(d)).$$

Both the probabilistic and the constructive approach use small dimensional acute sets as building blocks. So it is crucial for us to construct small dimensional acute sets of large cardinality. In the thesis we present an acute set of 8 points in $\mathbb{R}^4$ and an acute set of 12 points in $\mathbb{R}^5$ (disproving the conjecture of Danzer and Grünbaum for $d \geq 4$ already). We used computer to find acute sets in dimension $6 \leq d \leq 10$.

As far as $\kappa(d)$ is concerned, in large dimension (2) is still the best known lower bound. Bevan used computer to determine the exact values of $\kappa(d)$ for $d \leq 9$ [2]. He also gave a recursive construction improving upon the random constructions in low dimension. Our constructive approach yields a lower bound not only for $\alpha(d)$ but also for $\kappa(d)$, which further improves the bounds of Bevan in low dimension.

The following notion plays an important role in both approaches.

Definition 3.2. A triple A, B, C of three points in $\mathbb{R}^d$ will be called *bad* if for each integer $1 \leq i \leq d$ the i-th coordinate of B equals the i-th coordinate of A or C.

We denote by $\kappa_n(d)$ the maximal size of a set $S \subset \{0, 1, \ldots, n-1\}^d$ that contains no *bad triples*. It is easy to see that $\kappa_2(d) = \kappa(d)$ but our main motivation to investigate $\kappa_n(d)$ is that we can use sets without bad triples to construct acute sets recursively. We give an upper bound and two different lower bounds for $\kappa_n(d)$.

Theorem 3.3. *For even d*

$$\kappa_n(d) \leq 2n^{d/2},$$

and for odd d

$$\kappa_n(d) \leq n^{(d+1)/2} + n^{(d-1)/2}.$$

Theorem 3.4.

$$\kappa_n(d) > \frac{1}{2}\left(\frac{n^2}{2n-1}\right)^{\frac{d}{2}} > \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{d}{2}} = \left(\frac{1}{2}\right)^{\frac{d+2}{2}} n^{\frac{d}{2}}.$$

Theorem 3.5. *If $d \geq 2$ is an integer and $n \geq d$ is a prime power, then*

$$\kappa_n(d) \geq n^{\lceil \frac{d}{2} \rceil}.$$

Setting $n = 2$ and using that $\kappa_2(d) = \kappa(d)$ the next corollary readily follows from Theorem 3.3.

Corollary 3.6. *For even d*

$$\kappa(d) \leq 2^{(d+2)/2} = 2\left(\sqrt{2}\right)^d,$$

and for odd d

$$\kappa(d) \leq 2^{(d+1)/2} + 2^{(d-1)/2} = \frac{3}{\sqrt{2}}\left(\sqrt{2}\right)^d.$$

This corollary improves the upper bound $\sqrt{2}(\sqrt{3})^d$ given by Erdős and Füredi in [4].

4. The Koch curve is tube-null

Theorem 4.1 answers the following question posed by, among others, Marianna Csörnyei: is the Koch snowflake curve tube-null?

Theorem 4.1. *The Koch curve K is tube-null, that is, it can be covered by strips of arbitrarily small total width.*

Moreover, there exists a decomposition $K = K_0 \cup K_1 \cup K_2$ and projections π_0, π_1, π_2 such that the Hausdorff dimension of $\pi_i(K_i)$ is less than 1 for $i = 0, 1, 2$.

The proof contains geometric, combinatorial, algebraic and probabilistic arguments.

The thesis is based on the following papers

[H1] V. Harangi, Large dimensional sets not containing a given angle, *Cent. Eur. J. Math.*, to appear.

[H2] V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila, B. Strenner, How large dimension guarantees a given angle?, arXiv:1101.1426.

[H3] V. Harangi, Acute sets in Euclidean spaces, submitted.

[H4] V. Harangi, The Koch snowflake curve is tube-null, *Proc. Amer. Math. Soc.* **139** (2011), 1375–1381.

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