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Summary of Ph.D. Thesis

Measure and Category in Real Analysis

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1 Preface

In the Thesis we study three problems of real analysis in which the concepts of measure and category play an important role. The definitions and results related to these problems will be presented in three separate sections below.

The results of Section 2 are joint work with Z. Buczolich and M. Elekes, and the results of Section 4 are joint work with M. Elekes.

2 The topological Hausdorff dimension

We introduce a new concept of dimension for metric spaces, the so called *topological Hausdorff dimension*. We examine the basic properties of this new notion of dimension, compare it to other well-known notions and determine its value for some classical fractals. As our first application, we generalize the celebrated result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. As our second application, we show that the topological Hausdorff dimension is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space. We use various methods there, first of all the theory of random fractals and Baire category theory.

Let us recall the definition of the (small inductive) topological dimension.

Definition. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Our new dimension will be defined analogously, however, note that this second definition will not be inductive, and also that it can attain non-integer values as well. The Hausdorff dimension of X is denoted by $\dim_H X$ and let $\dim_H \emptyset = -1$.

Definition. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

2.1 Results

2.1.1 Basic properties of the topological Hausdorff dimension

Fact 2.1. $\dim_{tH} X = 0 \Longleftrightarrow \dim_t X = 0$.

The value of the topological Hausdorff dimension is always between the topological dimension and the Hausdorff dimension, in particular, this new dimension is a non-trivial lower estimate for the Hausdorff dimension.

Theorem 2.4. *For every metric space* X

$$\dim_t X \leq \dim_{tH} X \leq \dim_H X.$$

The topological Hausdorff dimension is an extension of the classical dimension, and there are some elementary properties one expects from a notion of dimension.

Fact 2.6 (Monotonicity)**.** *If* $X \subseteq Y$ *are metric spaces then* $\dim_{tH} X \leq \dim_{tH} Y$*.*

Corollary 2.8 (Bi-Lipschitz invariance)**.** *Let* X, Y *be metric spaces. If* $f \colon X \to Y$ *is bi-Lipschitz then* $\dim_{tH} X = \dim_{tH} Y$*.*

Theorem 2.16 (Countable stability for closed sets)**.** *Let* X *be a separable metric space and* $X = \bigcup_{n\in\mathbb{N}} X_n$*, where* X_n $(n \in \mathbb{N})$ *are closed subsets of* X*. Then* $\dim_{tH} X = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$*.*

The next theorem provides a large class of sets for which the topological Hausdorff dimension and the Hausdorff dimension coincide.

Theorem 2.19 (Products)**.** *Let* X *be a non-empty separable metric space. Then*

$$\dim_{tH} (X \times [0,1]) = \dim_H (X \times [0,1]) = \dim_H X + 1.$$

We give a complete description of the possible triples $(\dim_t X, \dim_{tH} X, \dim_H X)$ in Theorem 2.23, its consequence is the following.

Corollary 2.24. $\dim_{tH} X$ *cannot be calculated form* $\dim_t X$ *and* $\dim_H X$*, even for compact metric spaces.*

2.1.2 Calculating the topological Hausdorff dimension

We examine the values of the topological Hausdorff dimension of some classical fractals.

Theorem 2.25. *Let S be the Sierpiński triangle. Then* $\dim_{tH}(S) = 1$.

Fact 2.26. *If K is homeomorphic to* $[0,1]$ *then* $\dim_{tH} K = 1$.

Corollary 2.27. *Let K be the von Koch curve. Then* $\dim_{tH} K = 1$.

Theorem 2.28. *Let T be the Sierpiński carpet. Then* $\dim_{tH}(T) = \frac{\log 6}{\log 3}$.

We also consider Kakeya sets. Unfortunately, our methods do not give any useful information concerning the Kakeya Conjecture.

Theorem 2.30. *There exists a Kakeya set $K \subseteq \mathbb{R}^d$ of topological Hausdorff dimension 1 for every integer $d \geq 1$.*

We show that the trail of the Brownian motion almost surely (i.e. with probability 1) has topological Hausdorff dimension 1 in every dimension except perhaps 2 and 3. These two cases remain the most intriguing open problems of the chapter.

Problem 2.32. *Let $d = 2$ or 3. Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion.*

2.1.3 Application I: Mandelbrot's fractal percolation process

We generalize a result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. This limit set $M = M^{(p,n)}$ is a random Cantor set, which is constructed by dividing the unit square into $n \times n$ equal subsquares and keeping each of them independently with probability p, and then repeating the same procedure recursively for every subsquare.

Theorem (Chayes-Chayes-Durrett, [4])**.** *There exists a critical probability $p_c = p_c^{(n)} \in (0,1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.*

It is easy to see that this theorem is a special case of our next result. The above theorem essentially says that certain curves show up at the critical probability, and our proof shows that even 'thick' families of curves show up, which roughly speaking means a Lipschitz copy of $C \times [0,1]$ with $\dim_H C > d - 1$.

Theorem 2.34. *For every $d \in [0,2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0,1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \leq d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).*

We also give a numerical upper bound for $\dim_{tH} M$.

Theorem 2.49. *If $p > \frac{1}{\sqrt{n}}$ then almost surely*

$$\dim_{tH} M \leq 2 + 2\frac{\log p}{\log n}.$$

Corollary 2.50. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

2.1.4 Application II: The Hausdorff dimension of the level sets of the generic continuous function

B. Kirchheim [6] proved that for the generic continuous function f (in the sense of Baire category) defined on $[0,1]^d$, for every $y \in \operatorname{int} f\left([0,1]^d\right)$ we have $\dim_H f^{-1}(y) = d-1$, that is, as one would expect, 'most' level sets are of Hausdorff dimension $d-1$.

If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, thus every non-empty level set is of Hausdorff dimension 0.

If $\dim_t K > 0$ then the next theorem generalizes Kirchheim's result in a sense to compact metric spaces in place of $[0,1]^d$.

Theorem 2.71. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

(i) $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every $y \in \mathbb{R}$,*

(ii) for every $\varepsilon > 0$ there exists a non-degenerate interval $I_{f,\varepsilon}$ such that $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon$ *for every $y \in I_{f,\varepsilon}$.*

This immediately implies the following.

Corollary 2.72. *If K is a compact metric space with $\dim_t K > 0$ then we have* $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ *for the generic $f \in C(K)$.*

If K is also sufficiently homogeneous, for example self-similar then we can actually say more. Stand $B(x,r)$ for the closed ball of radius r centered at x.

Theorem 2.73. *Let K be a compact metric space with $\dim_t K > 0$, and assume that* $\dim_{tH} B(x,r) = \dim_{tH} K$ *for every $x \in K$ and $r > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

In the course of the proofs, as a spin-off, we also provide a sequence of equivalent definitions of $\dim_{tH} K$ for compact metric spaces. Perhaps the most interesting one is the following.

Corollary 2.64. *If K is a compact metric space then $\dim_{tH} K$ is the smallest number d for which K can be covered by a finite family of compact sets of arbitrarily small diameter such that the set of points that are covered more than once has Hausdorff dimension at most $d-1$.*

It can actually also be shown that in the equation $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ (for the generic $f \in C(K)$) the supremum is attained. On the other hand, one cannot say more in a sense, since there is a K such that for the generic $f \in C(K)$ there is a *unique* $y \in \mathbb{R}$ for which $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. However, for self-similar spaces we can replace 'the generic $y \in f(K)$' with 'for every $y \in \operatorname{int} f(K)$' as in Kirchheim's theorem. The results of this last paragraph are not included in the Thesis, see [B3] for them.

The results of the chapter are from [B2].

3 Duality in LCA Polish groups

In this chapter we study a problem concerning duality between measure and category. Let G be a locally compact abelian (LCA) Polish group. Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and null (with respect to Haar measure) subsets of G.

Definition. A bijection $F\colon G \to G$ is called an *Erdős-Sierpiński mapping* if

$$X \in \mathcal{N} \Leftrightarrow F[X] \in \mathcal{M} \quad \text{and} \quad X \in \mathcal{M} \Leftrightarrow F[X] \in \mathcal{N}.$$

Theorem (Erdős–Sierpiński)**.** *Assume the Continuum Hypothesis. Then there exists an Erdős–Sierpiński mapping on $\mathbb{R}$.*

The existence of such a function is independent from ZFC. Our main question is the following:

Is it consistent that there is an Erdős–Sierpiński mapping F that preserves addition, namely

$$\forall x, y \in G \quad F(x+y) = F(x) + F(y)?$$

This question is attributed to Ryll-Nardzewski in the case $G = \mathbb{R}$. Besides intrinsic interest, another motivation was the following: If this statement were consistent then the so called strong measure zero and strongly meager sets would consistently form isomorphic ideals. (For the definitions see [3].)

First T. Bartoszyński [1] gave a negative answer to the question in the case $G = 2^\omega$, then M. Kysiak proved this for $G = \mathbb{R}$ and answered the question of Ryll-Nardzewski, see [7], where he used and improved Bartoszyński's idea.

3.1 Results

We answer the general case, our aim is to prove the following theorem.

Theorem 3.1. *There is no addition preserving Erdős–Sierpiński mapping on any uncountable locally compact abelian Polish group.*

Let $(\varphi_\mathcal{M})$ denote the following statement (considered by Carlson in [3]): For every $S \in \mathcal{M}$ there exists a set $S' \in \mathcal{M}$ such that

$$\forall x_1, x_2 \in G \, \exists x \in G \quad (S + x_1) \cup (S + x_2) \subseteq S' + x.$$

Let $(\varphi_\mathcal{N})$ be the dual statement obtained by replacing $\mathcal{M}$ by $\mathcal{N}$.

If there exists an Erdős–Sierpiński mapping preserving addition then $(\varphi_\mathcal{M})$ and $(\varphi_\mathcal{N})$ are equivalent. On the one hand, $(\varphi_\mathcal{M})$ holds in LCA Polish groups. As the known proofs only work for the reals, we had to come up with a new, topological proof. On the other hand, we show that $(\varphi_\mathcal{N})$ fails for all uncountable LCA Polish groups by reducing the general case to three special cases. We follow the strategy developed in [5], we use structure theorems about LCA groups. At the end of the chapter we settle these three special cases, the proof is based on Kysiak's paper [7].

The results of the chapter are from [B1].

4 The structure of rigid functions

An easy calculation shows that the exponential function $f(x) = e^x$ has the somewhat 'paradoxical' property that $cf(x)$ is a translate of $f(x)$ for every $c > 0$. It is also easy to see that every function of the form $a + be^{kx}$ has this property. This connection is also of interest from the point of view of functional equations. In [2] Cain, Clark and Rose introduced the notion of vertical rigidity, which we now formulate for functions of several variables.

Definition. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is called *vertically rigid*, if $\operatorname{graph}(cf)$ is isometric to $\operatorname{graph}(f)$ for all $c \in (0, \infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

Obviously every function of the form $a + bx$ is also vertically rigid. D. Janković conjectured (see [2]) that the converse is also true for continuous functions.

Conjecture (D. Janković)**.** *A continuous function $f: \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

Cain, Clark and Rose [2] asked the following question.

Question 1 (Cain, Clark and Rose)**.** *Is every vertically rigid function $f: \mathbb{R} \to \mathbb{R}$ of the form $a + bx$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g?*

Janković's conjecture characterizes the continuous vertically rigid functions of one variable, so the following question is quite natural.

Question 2. *What are the continuous vertically rigid functions if there are more than one variables?*

Cain, Clark and Rose [2] introduce the following definition.

Definition. A function $f: \mathbb{R}^d \to \mathbb{R}$ is *horizontally rigid*, if graph $(f\,(c\vec{x}))$ is isometric to graph $(f\,(\vec{x}))$ for all $c \in (0, \infty)$.

Question 3. *Is every continuous horizontally rigid function $f: \mathbb{R} \to \mathbb{R}$ of the form $a + bx$?*

4.1 Results

Our main results are the characterizations of the continuous vertically rigid functions in one and two dimensions. First we study the one-dimensional case. We will need the following technical generalizations.

Definition. If C is a subset of $(0, \infty)$ and $\mathcal{G}$ is a set of isometries of $\mathbb{R}^{d+1}$ then we say that f is vertically rigid *for a set C via elements of $\mathcal{G}$* if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi\,(\mathrm{graph}(f)) = \mathrm{graph}(cf)$. (If we do not mention C or $\mathcal{G}$ then C is $(0, \infty)$ and $\mathcal{G}$ is the set of all isometries.)

We prove Janković's conjecture, even if we only assume that f is a continuous vertically rigid function for an uncountable set C.

Theorem 4.4 (Proof of Janković's conjecture)**.** *A continuous function $f: \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

For a function f let S_f be the set of directions between pairs of points on the graph of f, that is, let

$$S_f = \left\{ \frac{p-q}{|p-q|} : p, q \in \operatorname{graph}(f),\ p \neq q \right\}.$$

The proof of Theorem 4.4 is based on the observation that the S_{cf}'s ($c > 0$) are isometric if f is vertically rigid. We can reduce the problem to the case of translations, and we complete this case by solving function equations via algebraic methods.

If the isometries are translations, then we prove the following stronger theorem.

Theorem 4.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subseteq (0, \infty)$* via translations*. If f has a point of continuity then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$. If f is vertically rigid* via translations *(i.e. $C = (0, \infty)$) and bounded on a non-degenerate interval then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$, too.*

We show that Janković's conjecture fails for Borel measurable functions. Our counterexample also answers Question 1 in the negative.

Theorem 4.14. *There exists a Borel measurable vertically rigid function $f : \mathbb{R} \to [0, \infty)$ (via horizontal translations) that is not of the form $a + bx$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g.*

The function of Theorem 4.14 is zero almost everywhere (on a co-meager set). Therefore it is still possible that the complete analogue of Janković's conjecture holds: every vertically rigid Lebesgue (Baire) measurable function is of the form $a + bx$ or $a + be^{kx}$ *almost everywhere (on a co-meager set)*. We prove this in the case of translations.

Theorem 4.15. *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subseteq (0, \infty)$* via translations*. If f is Lebesgue (Baire) measurable then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$ almost everywhere (on a co-meager set).*

We also prove that in many situations the exceptional set can be removed. We use the methods of geometric measure (Baire category) theory.

Theorem 4.16. *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function that is of the form $a + bx$ $(b \neq 0)$ or $a + be^{kx}$ $(bk \neq 0)$ almost everywhere (on a co-meager set). Then f is of this form everywhere.*

If $f : \mathbb{R} \to \mathbb{R}$ is vertically rigid, then S_f is *rigid* according to the following definition.

Definition. For $c > 0$ let $\psi_c : \mathbb{S}^1 \to \mathbb{S}^1$ be the map obtained by 'multiplying by c', that is, let

$$\psi_c((x, y)) = \frac{(x, cy)}{|(x, cy)|} \quad ((x, y) \in \mathbb{S}^1).$$

We call a symmetric (about the origin) set $H \subseteq \mathbb{S}^1$ *rigid* for a set $C \subseteq (0, \infty)$ if for every $c \in C$ the sets $\psi_c(H)$ and H are isometric.

If f is Borel, then S_f is analytic, thus has the Baire property. Hence the following theorem can be considered as the first step towards handling vertically rigid Borel functions with general isometries.

Theorem 4.21. *Let $H \subseteq \mathbb{S}^1$ be a Baire measurable set that is rigid for an uncountable set C. Then in each of the four quarters of $\mathbb{S}^1$ determined by $(0, \pm 1)$ and $(\pm 1, 0)$ either H or $\mathbb{S}^1 \setminus H$ is meagre.*

We characterize the continuous vertically rigid functions of two variables, so we give a partial answer for Question 2.

Theorem 4.22. *A continuous function $f \colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if after a suitable rotation around the z-axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a, b, d, k \in \mathbb{R},\ k \neq 0,\ s \colon \mathbb{R} \to \mathbb{R}$ continuous).*

The proof is based on similar ideas as Theorem 4.4, but handling the various cases and technical difficulties are much more complicated.

We answer Question 3 in the case of translations, the following holds.

Theorem 4.38. *A function $f \colon \mathbb{R} \to \mathbb{R}$ is horizontally rigid* via translations *if and only if there exists $r \in \mathbb{R}$ such that f is constant on $(-\infty, r)$ and (r, ∞).*

Later C. Richter answered Question 3, he showed in [8] that every continuous horizontally rigid $f \colon \mathbb{R} \to \mathbb{R}$ function is affine. M. Elekes and the author of this thesis proved in [B4] that this is also true in dimension 2. If there are more than two variables, we do not have even conjectures about the characterization of continuous vertically rigid functions, and we also cannot decide whether every continuous horizontally rigid function is affine.

The results of the chapter are from [B6] and [B5].

The Ph.D. Thesis is based on the following papers

[B1] R. Balka, *Duality between measure and category in uncountable locally compact abelian Polish groups*, to appear in Real Anal. Exchange.

[B2] R. Balka, Z. Buczolich and M. Elekes, *A new fractal dimension: The topological Hausdorff dimension*, submitted.

[B3] R. Balka, Z. Buczolich and M. Elekes, *Topological Hausdorff dimension and level sets of generic continuous functions on fractals*, submitted.

[B4] R. Balka and M. Elekes, *Continuous horizontally rigid functions of two variables are affine*, submitted.

[B5] R. Balka and M. Elekes, *The structure of continuous rigid functions of two variables*, Real Anal. Exchange **35**, no. 1 (2009), 139–156.

[B6] R. Balka and M. Elekes, *The structure of rigid functions*, J. Math. Anal. Appl. **345**, no. 2 (2008), 880–888.

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