Dido Theorem and Other Problems in Planes of Constant Curvature

Theses of Doctoral Dissertation

By Kertész, Gábor

PHD School of Mathematics
Doctoral Program of Theoretical Mathematics

Head of School: Dr. Laczkovich, Miklós
Head of Program: Dr. Szűcs András
Supervisor: Dr. Böröczky, Károly

Eötvös Loránd University
Faculty of Natural Sciences
2011.

Introduction

The main part of the dissertation is the examination the Dido problem put by L. Fejes-Tóth in the 1960-s not only in the euclidean but also in the hyperbic plane. The name of the problem is from the name of the legendary queen Dido. Dido was given as much land as she could bounded by the leather of an ox. The queen made tiny strips tied them together and such a way bounded a huge part of the seaside by a half circle. And so was founded Carthago by Vergil. L. Fejes-Tóth László put the question [2], what would have done Dido without the sea and with a given collection of rods? The conjecture is plausibke. If there is a chord polygon with the given sides and the polygon contains the centre of the circumscribed circle, the that is the maximum possible bounded by the rods. If not, then you should shorten the longest rod as far as it is a diameter of the circumscribed circle.

At glance one can not see what is the difficulty, as it is easy to see, that among the n-gons with given side lengths the chord polygon mentioned above is the optimal one. The problem is, that the segments may crossing in a fuzzy way. We feel, that it wastes, but the estimation of the area of bounded set is very dificult. More good mathematicians tried to prove the conjecture without success. After more than twenty years I succeeded to prove it in my MCS thesis. Later A. Siegel showed much shorter and striking proof [4].

What is the reason of my reprove? Siegel used typical euclidean calculus, while my method can be extended for the hyperbolic plane. I really succeeded to prove the Dido conjecture for the hyperbolic plane as you can see it by reading the second unit.

It is well known in the euclidean plane, that with given side lengths the chord polygon is of maximum area. It is much less known, that in the spherical and hyperbolic case the polygon inscribed in a cycle (circle, paracycle os hypercycle) is optimal. The proof of the second statement [1] based on the fact, that a closed curve of given perimeter containing a given segment bounds the domain of maximum area, if a cycle arch connects the ends of the segment [5]. As I love elementary argumentations, I show such a proof of the polygonal Dido theorem in the first unit. In addition this proof is unified in the planes of constant curvature.

The theme of the third unit is an independent one, but it is very nice. A. Heppes put the question [3], that for given n which is the

minimal d_n, such that you can cover the unit sphere with n sets of at most d_n diameters? He veryfied, that $d_8 = 90°$, and guessed $d_9 = d_8$. His argumentation proved to be false, so there was the given the possibility to show a right one in the last unit.

Theses

The first unit proves statements true in planes of constant curvature. None of them are new results, but I give a very simple elementary proof for their base lemma, named chord qudrangle lemma. In the next statements the word cycle means a circle, a paracycle or a hypercycle.

THEOREM 1 (Polygonal Dido). *Among the n-gons of side lengths at most $l_1,\ldots,l_n$ there is an only n-gon H of maximum area. For this H one of the next two statements holds:*

(1) There is a circumscribed circle around H, H contains the centre of this circle and the side lengths of H equal the given numbers.

(2) There is a circumscribed circle around H, the longest side of H is a diameter of the circle and all the other side lengths of H equal the given numbers.

This theorem proved similarily to the proof of the same theorem in the paper of A. Siegel [4] in the euclidean plane.

THEOREM 2 (Joint polygon). *Among the n-gons of side lengths $l_1,\ldots,l_n$ the n-gon H inscribed into a cycle is of maximum area. In 2-sphere we demand that the perimeter is smaller than π.*

You can prove this theorem for the next chord quadrangle lemma with ease.

LEMMA 1 (Chord quadrangle). *Among the quadrangles with given side lengths the one inscribed into a cycle is of maximum area. In 2-sphere we demand that the perimeter is smaller than π.*

The heart of the proof is the next

LEMMA 2 (Symmetry). *Among the konvex $ABCD$ quadrangles with side lengths $AB = a$, $BC = DA = b$ és $CD = c$ the one inscribed into a cycle is of maximum area.*

The main result of the next unit is the Dido conjecture of L. Fejes-Tóth.

A finite collection of positive numbers $L(l_1,\ldots,l_n)$ is given. If in an $S(s_1,\ldots,s_n)$ collection of segments the length of s_i equals l_i, we say that S have sizes L.

The segments of S divides the (euclidean, hyperbolic or spherical) plane into polygonally connected parts. The union of the bounded

parts (or the ones smaller than a hemisphere) is the point set bounded by S.

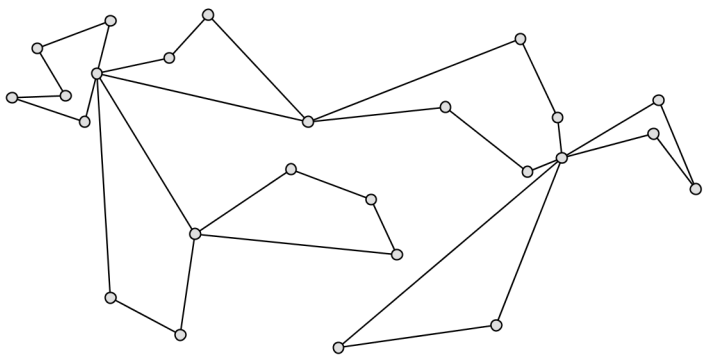
Now we put down the next

THEOREM 3 (Dido). *In the euclidean or hyperbolic plane the collection S of segments with sizes L bounds the set H of maximum area if one of the next statements hold:*

(1) There is a circumscribed circle around H, H contains the centre of this circle and the sides of H are the whole segments of S.
(2) There is a circumscribed circle around H, the longest segment is shortened to be a diameter of the circle and all the other sides are the other whole segments of L.

My proof uses the notion of t-polygons and painted t-polygon:

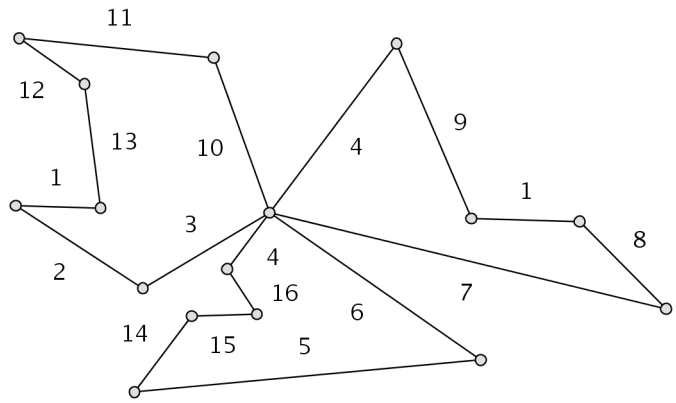
DEFINITION 1 (t-polygon). *The polygonally connected union of polygons pairweise at most one point in common is called t-polygon, if the set bounded by the sides of polygons is the union of the polygonal discs and the common point of two polygons is a vertex of both polygons.*



DEFINITION 2 (Painted t-polygon). *Assign natural number called color to each side of the S t-polygon such a way, that the segments of same colors are subsegments of a segment in the clesed disc of t-polygon. The t-polygon with this assignment is called painted t-polygon.*

Note the color set of the painted t-polygon S by $C(S)$-sel, the total lengths of the sides of the same color c by $l(c)$-sel, and the color of side e in S by $c(e)$. The count of sides in S with color $c \in C(S)$ is the ordo of c denoted by $o(c)$. The ordo $o(c(e))$ of the $c(e)$ color of side e is the ordo of side e and denoted by $o(e)$.

The Dido theorem is the consequence of the most essential



THEOREM 4 (Painted t-polygon). *The area of painted t-polygon F is at most the area of a chord polygon H with the side lengths $l(c)$, $c \in C(F)$ in any order. Equality holds iff F is a chord polygon with different colors for different sides.*

The third unit contains a proof of a statement of A. Heppes. He launched a very nice problem in his paper [3]. He put the question, that for a positive natural n what is the least possible d_n, for which you can cover the unit sphere with n sets of diameters at most d_n. One can see with ease, that $d_1 = d_2 = d_3 = \pi$, és $d_4 = cos^{-1}(-\frac{1}{6})$. Heppes determined d_n for $n \leq 6$ and for $n = 8$. He stated, that surprisingly $d_9 = d_8$. There was an error in the argumentations, and encouraged us to examine the problem.

I succeeded to prove, that the conjecture of A: Heppes is true, ie

THEOREM 5. *If $d < \frac{\pi}{2}$, then you can not cover the unit sphere by nine sets of diameters at most d.*

Bibliography

[1] Károly Böröczky. személyes beszélgetés.

[2] L. Fejes-Tóth. "Über das Didosche Problem". In: *Elem. Math.* 23 (1968), pp. 97–101.

[3] A. Heppes. "Decomposing the 2-sphere into domains of smallest possible diameter". In: *Periodica Math. Hungarica* 36/2-3 (1998), pp. 171–180.

[4] Alan Siegel. "A Dido Problem as modernized by Fejes-Tóth". In: *Discrete & Computational Geometry* 22 (2002), pp. 227–238.

[5] V. A. Zalgaller Yu. D. Burago. *Geometric inequalities*. Springer-Verlag.