Additive representation functions

Doctoral theses

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1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, and let $k \geq 2$ be a fixed integer. Let $\mathcal{A} = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of positive integers. For $k \geq 2$ integer and $\mathcal{A} \subset \mathbb{N}$, and for $n = 0, 1, 2, \dots$ let $R_1(\mathcal{A}, n, k)$, $R_2(\mathcal{A}, n, k)$, $R_3(\mathcal{A}, n, k)$ denote the number of solutions of the equations

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A},$$

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A}, \quad a_{i_1} < a_{i_2} < \dots < a_{i_k},$$

and

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A}, \quad a_{i_1} \leq a_{i_2} \leq \dots \leq a_{i_k},$$

respectively. If $F(n) = O(G(n))$ then we write $F(n) << G(n)$. Put

$$A(n) = \sum_{\substack{a \in \mathcal{A} \\ a \leq n}} 1.$$

The research of the additive representation functions began in the 1950's. Starting from a problem of Sidon, P. Erdős proved that there exists a sequence $\mathcal{A} \in \mathbb{N}$ so that there are two constans c_1 and c_2 for which for every n

$$c_1 \log n < R_1(\mathcal{A}, n, 2) < c_2 \log n.$$

On the other hand an old conjecture of Erdős states that for no sequence $\mathcal{A}$ can we have

$$\frac{R_1(\mathcal{A}, n, 2)}{\log n} \to c \quad (0 < c < +\infty).$$

There are some related questions in [3] and [12]. These problems led P. Erdős, A. Sárközy and V. T. Sós to study the regularity property and the monotonicity of the function $R_1(\mathcal{A}, n, 2)$ see in [6], [7], [8], [9]. In my thesis I study the regularity properties and the monotonicity of the representation function $R_1(\mathcal{A}, n, k)$ for $k > 2$ integer. I extend and generalize some result of P. Erdős, A. Sárközy and V. T. Sós by using the generator function method and the probabilistic method.

2 The methods we are working with

In my thesis I use the generating function method. We start out from the generating function of the sequence $\mathcal{A}$:

$$f(z) = \sum_{a \in \mathcal{A}} z^a.$$

It is easy to see that

$$f^k(z) = \sum_{n=1}^{\infty} R_1(\mathcal{A}, n, k) z^n.$$

We use the generating function method to prove the results about the monotonicity. We also use the Hölder - inequality, the Cauchy - inequality and the Parseval - formula. In the next step I tell a few words about the probabilistic method. An important problem in additive number theory is to prove that a sequene with certain properties exists. One of the essential ways to obtain an affirmative answer for such a problem is to use the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the Halberstam - Roth book [12]. To show that a sequence with a property $\mathcal{P}$ exists, it sufficies to show that a properly defined random sequence satisfies $\mathcal{P}$ with positive probability. Usually the property $\mathcal{P}$ requires that for all sufficiently large $n \in \mathbb{N}$, some relation $\mathcal{P}(n)$ holds. The general strategy to handle this situation is the following. For each n one first shows that $\mathcal{P}(n)$ fails with a small probability, say p_n. If p_n is sufficiently small so that $\sum_{n=1}^{+\infty} p_n$ converges, then by the Borel - Cantelli lemma, $\mathcal{P}(n)$ holds for all sufficiently large n with probability 1. Note that in my proofs the additive representation function is the sum of random variables. However for $k > 2$ these variables are not independent. To overcome this trouble in my thesis I apply the theorems of J. H. Kim and V. H. Vu, [15], [25], [26], [27], the Janson inequality [14] and the method of Erdős and Tetali [2], [10].

3 Theses

For $i = 1, 2, 3$ we say $R_i(\mathcal{A}, n, k)$ is monotonous increasing in n from a certain point on, if there exists an integer n_0 with

$$R_i(\mathcal{A}, n+1, k) \geq R_i(\mathcal{A}, n, k) \quad for \quad n \geq n_0.$$

In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied the monotonicity properties of the three representation functions $R_1(\mathcal{A}, n, 2)$, $R_2(\mathcal{A}, n, 2)$, $R_3(\mathcal{A}, n, 2)$. A. Sárközy proposed the study of the monotonicity of the functions $R_i(\mathcal{A}, n, k)$ for $k > 2$ [2, Problem 5]. He conjectured [3, p. 337] that for any $k \geq 2$ integer, if $R_i(\mathcal{A}, n, k)$ $(i = 1, 2, 3)$ is monotonous increasing in n from a certain point on, then $A(n) = O(n^{2/k-\varepsilon})$ cannot hold. In this thesis I prove (see in [18]) the following slightly stronger result on $R_1(\mathcal{A}, n, k)$:

Theorem 1. *If $k \in \mathbb{N}$, $k \geq 2$, $\mathcal{A} \subset \mathbb{N}$ and $R_1(\mathcal{A}, n, k)$ is monotonous increasing in n from a certain point on, then*

$$A(n) = o\left(\frac{n^{2/k}}{(\log n)^{2/k}}\right)$$

cannot hold.

Let $k \geq 2$, $l \geq 1$ be a fixed integers. If $s_0, s_1, s_2 \ldots$ is a given sequence of real numbers, then let $\Delta_l s_n$ denote the l-th difference of the sequence

$s_0, s_1, s_2 \dots$ defined by $\Delta_1 s_n = s_{n+1} - s_n$ and $\Delta_l s_n = \Delta_1(\Delta_{l-1} s_n)$. It is well-known and it is easy to see by induction that

$$\Delta_l s_n = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} s_{n+i}.$$

Let $B(\mathcal{A}, N)$ denote the number of blocks formed by consecutive integers in $\mathcal{A}$ up to N, i.e.,

$$B(\mathcal{A}, N) = \sum_{\substack{a \le N \\ a-1 \notin \mathcal{A}, a \in \mathcal{A}}} 1.$$

P. Erdős, A. Sárközy and V. T. Sós studied the following problem: what condition is needed to ensure

$$\limsup_{n \to +\infty} |R_1(\mathcal{A}, n+1, 2) - R_1(\mathcal{A}, n, 2)| = +\infty?$$

They proved in [7] that if $k = 2$, $l = 1$ and if $\lim_{N \to \infty} \frac{B(\mathcal{A},N)}{\sqrt{N}} = \infty$, then the above holds. They also proved that their result is nearly sharp.
In [16] I extended their Theorem to any $k > 2$:

Theorem 2. *If $k \ge 2$ is an integer and $\lim_{N \to \infty} \frac{B(\mathcal{A},N)}{\sqrt[k]{N}} = \infty$, and $l \le k$, then $|\Delta_l R_1(\mathcal{A}, n, k)|$ cannot be bounded.*

I also proved in [20] that the above result is nearly best possible:

Theorem 3. *For all $\varepsilon > 0$, there exists an infinite sequence $\mathcal{A}$ such that*

(i) $B(\mathcal{A}, N) \gg N^{1/k-\varepsilon}$,

(ii) $R_1(\mathcal{A}, n, k)$ is bounded so that also $\Delta_l R_1(\mathcal{A}, n, k)$ is bounded if $l \le k$.

In the case $l > k$ I have only a partial result [17]:

Theorem 4. *If $l \ge 2$ an integer and $\lim_{N \to \infty} \frac{B(\mathcal{A},N)}{\sqrt{N}} = \infty$, then $|\Delta_l(R_1(\mathcal{A}, n, 2))|$ cannot be bounded.*

In [13] G. Horváth extended a theorem of Erdős and Sárközy to any $k > 2$ integer. In [19] I proved that his result is nearly best possible by using probabilistic method:

Theorem 5. *If $k > 2$ is a positive integer, c_8 is a constant large enough in terms k, $F(n)$ is an arithmetic function satisfying*

$$F(n) > c_8 \log n \quad for \quad n > n_0,$$

and there exists a real function $g(x)$, defined for $0 < x < +\infty$, and real numbers x_0, n_1 and c_7, c_9 costants such that

(i) $0 < g(x) \leq \frac{(\log x)^{\frac{1}{k}}}{x^{1-\frac{k+1}{k^2}}} < 1$ *for* $x \geq x_0$,

(ii) $\left|F(n) - k! \sum_{\substack{x_1+x_2+\ldots+x_k=n \\ 1\leq x_1<x_2<\ldots<x_k<n}} g(x_1)g(x_2)\ldots g(x_k)\right| < c_7(F(n)\log n)^{1/2}$ *for* $n > n_1$,

then there exists a sequence $\mathcal{A}$ such that

$$|R_1(\mathcal{A}, n, k) - F(n)| < c_9(F(n)\log n)^{1/2} \quad for \quad n > n_2.$$

We say a set $\mathcal{A}$ of positive integers is an asymptotic basis of order h if every large enough positive integer can be represented as the sum of h terms from $\mathcal{A}$. In other words $\mathcal{A}$ is an asymptotic bases of order h if there exists an n_0 positive integer such that $R_3(\mathcal{A}, n, 2) > 0$ for $n > n_0$. A set of positive integers $\mathcal{A}$ is called Sidon set if all the sums $a + b$ with $a \in \mathcal{A}$, $b \in \mathcal{A}$, $a \leq b$ are distinct. In other words $\mathcal{A}$ is a Sidon set if $R_3(\mathcal{A}, n, 2) \leq 1$. In [3] and [4] P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set which is an asymptotic basis of order 3. The problem also appears in [9] (with a typo in it: order 2 is written instead of order 3). In [6] G. Grekos, L. Haddad, C. Helou and J. Pihko proved that a Sidon set cannot be an asymptotic basis of order 2. Recently J. M. Deshouillers and A. Plagne in [1] constructed a Sidon set which is an asymptotic basis of order at most 7. In this thesis (see also in [21]) I improve on this result by proving:

Theorem 6. *There exists an asymptotic basis of order 5 which is a Sidon set.*

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