Cobordism of singular maps

Theses of PhD dissertation

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1 Introduction

Local singularity theory studies the properties of smooth $(\mathbb{R}^n, 0) \to (\mathbb{R}^{n+k}, 0)$ mapping germs. Two germs are called $\mathcal{A}$-equivalent if a smooth reparametrization of the source and the target space takes one to the other. The resulting equivalence classes are called (mono)singularities. Complete classification of singularities is currently a hopeless problem, here we will only investigate cases where the classification is already known.

Global singularity theory focuses on singular loci of mappings between smooth manifolds. From this point of view it becomes important to consider the local behaviour of a mapping on the full preimage of a point in the target space. Such a description, consisting of multiple monosingularities, is called a multisingularity.

The set of points at which a given sufficiently generic map has a certain monosingularity or multisingularity often carries information about other properties of the mapping itself as well as the source and target manifolds. The initial question of this thesis is the following problem: given a mapping, what is the obstruction to the existence of a mapping that is equivalent to it in some sense, but only has certain (multi)singularities?

In the study of mappings between manifolds we aim for classifications up to so-called singular bordism and singular cobordism. That is, we consider two mappings equivalent, if together they form the boundary of a mapping between manifolds with boundary that only has singularities from a set fixed in advance. The reason for this choice is that while the manifolds are well understood up to abstract cobordism (without restrictions on the involved mappings) thanks to the works of Thom and Wall, there is no such practically applicable classification for a finer equivalence relation, hence the investigation of mappings would have to be cumbersome.

2 Methods and results

The primary tool for investigation of bordism and cobordism groups is the generalized Pontryagin-Thom construction [9], which transforms the calculation of cobordism groups into a purely homotopy theoretical question. The investigation of this construction and the classifying spaces that play a fundamental role in it is the main goal of this thesis.

2.1 Cobordism groups

The general construction of classifying spaces [9] glues these spaces together from blocks corresponding to the allowed multisingularities. As a consequence, the homotopy groups of the resulting space are very hard to compute. Szűcs [16] proved that when the set of allowed multisingularities consists of all multisingularities composed from a fixed set of monosingularies (there are no global restrictions), there exists a so-called "key fibration" [16, Definition 109] between the classifying spaces that makes the calculation of their homotopy groups more approachable. We give a new, more geometric proof of the existence of the key fibration, which stays valid under more general conditions than the original one. For example, we can prove singularity removal theorems in the case of negative codimensional mappings analogous to the positive codimensional ones:

Theorem 1. *If M is a closed 4-manifold and P is a closed 3-manifold, then any smooth generic mapping $f : M \to P$ is cobordant to a generic mapping without definite or indefinite swallowtail singularities. Should M and P be given orientations, the source manifold of the cobordism can be chosen to be oriented as well.*

As another extension, we can handle a case with a global restriction (generalizing [16, Proposition 108]):

Theorem 2. *Let η be a monosingularity of $k > 0$ codimensional mappings,* *let τ' be the set of all multisingularities composed from a fixed set of monosingularities, and assume that while $\partial\eta \subseteq \tau'$, $\eta \notin \tau'$. Furthermore let τ_r for $r > 0$ be the set of all multisingularities composed from a singularity in τ' and at most r points with η singularity, and denote by $\tilde{\Gamma}_r$ the classifying space of immersions with a normal $\tilde{\xi}_\eta$-structure and at most r-tuple self-intersection points. Then the natural forgetful mapping $X_{\tau_r} \to \tilde{\Gamma}_r$ is a Serre fibration with fiber $X_{\tau'}$.*

Corollary 3. *Under the assumptions of the previous theorem a τ_r-mapping is τ_r-cobordant to a τ'-mapping exactly when its singular set of η-points is null-cobordant as an immersion with a normal $\tilde{\xi}_\eta$-structure and at most r-tuple self-intersection points.*

With the help of the key fibration we compute the cobordism groups of fold maps (which may only have the simplest monosingularity apart from the regular point) in the first two cases when they cannot be trivially identified with the abstract cobordism groups.

Theorem 4 ([20]). *Denote by τ the set of all multisingularities composed from regular and fold monosingularities.*

(a) *For all $k \geq 1$ $Cob_\tau(2k+1, k) \cong \mathfrak{N}_{2k+1}$;*

(b$_1$) *$Cob_\tau^{SO}(5, 2) \cong \Omega_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$;*

(b$_*$) *For all $m \geq 2$ $Cob_\tau^{SO}(4m+1, 2m) \cong \Omega_{4m+1}$;*

(c) *For all $m \geq 1$ $Cob_\tau^{SO}(4m-1, 2m-1) \cong \Omega_{4m-1} \oplus \mathbb{Z}_{3^t}$, where $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$ and $\alpha_3(x)$ denotes the sum of digits of the natural number x in base 3.*

Theorem 5. *a) $Cob_\tau(2k+2, k)$ is the kernel of the characteristic number $\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k$, an index 2 subgroup of $\mathfrak{N}_{2k+2}$;*

b$_2$) $Cob_\tau^{SO}(6, 2) \cong 0$;

b_*) for all even $k \geq 4$ the group $Cob_\tau^{SO}(2k+2, k)$ is the kernel of the characteristic number $\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k$, an index 2 subgroup of Ω_{2k+2};

c) for all odd k the group $Cob_\tau^{SO}(2k+2, k)$ is the kernel of the characteristic number $\bar{p}_{(k+1)/2}$, a nontrivial subgroup of Ω_{2k+2}.

In the first case a small modification of the argument gives the bordism groups of fold maps in relation to the bordism groups of all (oriented) maps [13].

Theorem 6. (a) *For all $k \geq 0$ $Bord_\tau(2k+1, k) \cong \mathfrak{N}(2k+1, k)$;*

(b) *For all $m \geq 1$ $Bord_\tau^{SO}(4m+1, 2m) \cong \Omega(4m+1, 2m)$;*

(c) *For all $m \geq 1$ the following sequence is exact:*

$$0 \to \mathbb{Z}_{3^u} \to Bord_\tau^{SO}(4m-1, 2m-1) \to \Omega(4m-1, 2m-1) \to 0;$$

here u satisfies the inequality $0 \leq u \leq t$, $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$, using the notation of Theorem 4.

In the course of the proof we give new examples of mappings that have a null-homologous cusp locus both in source and target (hence the Thom polynomial [17] cannot detect the cusp singularity), but no fold mapping is even abstractly cobordant to it.

Similar results can be proven in the case when τ is the set of all multisingularities composed from regular, fold and cusp points, and the goal is the elimination from the cobordisms of the so-called $III_{2,2}$ singularity, which is the simplest corank 2 monosingularity.

If the dimension of the singularity to avoid is more than 1, the geometric methods employed in the proofs of the theorems above fail. For relatively small dimensions, however, we can still assign geometric meaning to the obstructions that need to be computed. These obstructions are elements of the homotopy groups of the form $\pi_*(\Gamma T\tilde{\xi})$, where $\tilde{\xi}$ denotes the universal target bundle [9] of the singularity; it describes the global behaviour of the singular points in the target manifold. For a fixed r denote by $\mathcal{C}$ the Serre class of finite abelian groups of odd order divisible only by primes dividing $r+1$ and let $\mathcal{C}^+$ be the class of finite abelian groups of order divisible only by primes dividing $2(r+1)$. Isomorphism up to groups in $\mathcal{C}$ and $\mathcal{C}^+$ will be denoted by $\cong_{\mathcal{C}}$ and $\cong_{\mathcal{C}^+}$ respectively.

Theorem 7. *Let $\tilde{\xi}$ be the universal target bundle of the Morin singularity $\Sigma^{1_r,0}$ in either the unoriented or the oriented case. For a given mapping $\kappa : \mathbb{S}^{n+k} \to T\tilde{\xi}$, set $M = M(\kappa) = \kappa^{-1}(0_{\tilde{\xi}})$ the transverse preimage of the zero section of $\tilde{\xi}$ (with the induced orientation when $\tilde{\xi}$ is orientable). Then for all $0 \leq m < k$ the following statements hold.*

- *If $\tilde{\xi}$ is not orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \in \mathcal{C}^+$.*
- *If $\tilde{\xi}$ is orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}^+} \Omega_m$. The isomorphism is given by the abstract oriented bordism class of M.*
- *If r is even and $\tilde{\xi}$ is not orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \mathfrak{N}_m(\mathbb{R}P^\infty)$. The isomorphism is given by the cobordism class of M decorated with the kernel bundle of the mapping f classified by κ.*
- *If r is even and $\tilde{\xi}$ is orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \Omega_m(\mathbb{R}P^\infty)$. The isomorphism is given by the oriented cobordism class of M decorated with the kernel bundle.*

2.2 Bordism groups

The singular bordism groups are somewhat easier to compute than the singular cobordism groups, since the generalized Pontryagin-Thom construction identifies the former groups with the abstract bordism groups of the corresponding classifying spaces, and these can be handled even with the original block-construction. Additionally, when looking for cohomological obstructions, investigating the so-called Kazarian space instead of the much more complicated classifying space already gives results. For example, with the help of the Kazarian space we can determine the avoiding ideal of the cusp singularity. The avoiding ideal [4] (with $\mathbb{Z}_2$ coefficients) of the monosingularity η is the set of those Stiefel-Whitney characteristic classes that vanish on the virtual normal bundle of any smooth, fiberwise polynomial bundle mapping between vector bundles.

Theorem 8. *The avoiding ideal of the singularity $\Sigma^{1,1}$ is generated as an $H^*(BO;\mathbb{Z}_2)$ ideal by the set*

$$\{w_{k+l}w_{k+m}+w_{k+q}w_{k+r}|l,m,q,r\geq 0 \text{ and } l+m=q+r\geq 2\}.$$

Corollary 9. *The avoiding ideal of the singularity $\Sigma^{1,1}$ consists exactly of the classes of the form $\sum_{I\in\mathcal{I}} w_I$ such that*

$$\sum_{I\in\mathcal{I}} c^S w_k^{|I^+|} w_{I\setminus I^+}=0,$$

where $\mathcal{I}$ contains only index sets I with $\max I>k$, I^+ denotes the subsequence $\cup\{J\subseteq I|\min J>k\}$ and $S=\sum_{i\in I^+}(i-k)$.

As an application of this result we can obtain nearly optimal bounds on the existence of fold mappings of real projective spaces into Euclidean spaces:

Theorem 10. *Let $n=2^s+t$ with s and $t<2^s$ nonnegative integers. If there exists a fold mapping from $\mathbb{R}P^n$ to $\mathbb{R}^{n+k}$, then*

- *if $\frac{4}{3}2^s<n<2^{s+1}$, then $k\geq 2^{s+1}-n-2$.*
- *if $2^s<n<\frac{4}{3}2^s$, then*
    - *for $n=2^u(8a+3)+b$ with $0\leq b<2^u$ and maximal u, $k\geq 2^{u+2}a+2^u-b-2$.*
    - *if $\left\lfloor\frac{n}{2^u}\right\rfloor\not\equiv 3 \bmod 4$ for all $u\geq 0$, then*

* $k \geq \frac{n-3}{2}$ *for odd* n *and*
* $k \geq \frac{n}{2} - 2^{p-1}$ *for* $n = 2^p m$ *with an odd* m.

Using a deep result of [16], the so-called Kazarian conjecture, we compute the unoriented bordism groups of fold mappings from the homology groups of the Kazarian space of the fold mappings.

Finally we investigate the classifying space to obtain singular bordism groups in the case when allowed mappings have only single fold points and regular points with bounded multiplicity. For a fixed codimension $k > 0$ denote by I_2 the set containing the fold and the regular points with multiplicity at most 2; let q_m be the number of partitions of the nonnegative integer m that contain no elements greater than k, and set $d_m = \dim_{\mathbb{Z}_2} \mathfrak{N}_m$.

Theorem 11.

$$\dim Bord_{I_2}(n) = \sum_{s=0}^{n+k} \sum_{j=0}^{\left\lfloor \frac{n-k-s-1}{2} \right\rfloor} q_j q_{n-k-s-j} d_s + \sum_{s=0}^{n+k} \sum_{r=\left\lceil \frac{n-k-s}{2} \right\rceil}^{n-k-s-1} q_r d_s + \\ + \sum_{s=0}^{n+k} q_{n-s} d_s + d_{n+k}$$

We get a similar explicit formula for the I_2-bordism group of cooriented mappings, and from these we determine the rest of the unoriented, respectively cooriented I-bordism groups for all such sets I that contain only single fold points and regular points with bounded multiplicity. In the case of mappings between oriented manifolds we compute the I-bordism groups rationally.

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