PhD Thesis

Outline

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On the Stokes problem

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1 Introduction

This thesis is concerned with the Stokes problem, which originates from the Navier-Stokes equations [25] for the motion of a viscous fluid:

$$-\Delta\vec{u} + \operatorname{grad} p = \vec{f} \text{ and } \operatorname{div}\vec{u} = 0. \quad (1)$$

This equation system is to be solved for the unknown velocity $\vec{u}$ and the corresponding pressure p on a planar or spatial domain Ω in appropriate chosen function spaces if one prescribes boundary values for the velocity along the boundary of the domain. The Stokes problem is called of first kind if one prescribes Dirichlet boundary conditions for the velocity.

The thesis contains two main sections.

In the first section we investigate several properties of the Schur complement operator of first kind Stokes problem posed on a plane domain with the help of conformal mapping. Our aim is to show the possible utilization of conformal mapping in reproving already known results and in achieving new ones. We also examine the inf-sup constant which is a number depending only on the shape of the problem domain having a decisive role for the stable solvability of the Stokes problem.

In the second section we give results on the representation of Stokes functions, which are the velocity solutions of the homogeneous Stokes equations (i.e. $\vec{f} = 0$ in (1)) for prescribed Dirichlet boundary conditions. We omit the complex formalism, because we concern also three-dimensional domains. We also generalize these representation formulae to the case of Naviers equation in linear elasticity.

The first section of the thesis is mainly based on my two published papers [53] and [54], while the second section is based on my third paper [55].

2 Results via conformal mapping

In this section we examine mainly the Schur complement operator

$$\mathcal{S} = \operatorname{div}\Delta_0^{-1}\operatorname{grad} : L_2(\Omega) \to L_2(\Omega), \quad (2)$$

of the first kind Stokes problem, where Δ_0 denotes the vector Laplace operator corresponding to homogeneous Dirichlet boundary values on the boundary of the domain Ω.

We reprove in our implementation the connection between $\mathcal{S}$ and the Friedrichs operator of the domain Ω, which is defined by

$$\mathcal{F} = \mathcal{P} \circ \mathcal{C} : AL_2(\Omega) \to AL_2(\Omega), \quad (3)$$

where $\mathcal{P}$ is the orthogonal projection of $L_2(\Omega)$ onto the subspace $AL_2(\Omega)$ containing analytic functions, and where $\mathcal{C}$ denotes the conjugacy operator.

This connection, already announced in [13], is formulated in

Theorem 2.1 *Let Ω be a plane domain with enough smooth boundary. Then we have*

$$2\mathcal{S} = \mathcal{I} - \mathcal{C} \circ \mathcal{F} \tag{4}$$

for the Friedrichs operator $\mathcal{F}$ and the Schur complement operator $\mathcal{S}$ of the domain. □

Remark 2.2 The smoothness of the boundary $\partial\Omega$ of the domain Ω, which plays a decisive role in Theorem 2.1, can be characterized for a simply connected domain with the help of its corresponding Riemann mapping function [37], which maps the domain conformal onto the unit disc D. Theorem 2.1 is also proved for multiply connected domains with enough smooth boundary using a result of [22]. □

The usage of conformal mapping allows us to give a certain matrix representation for the studied operators (2) and (3). First we observe that

$$\mathcal{F}_D(p) := (\mathcal{F}(f) \circ g)g' \text{ for } p = (f \circ g)g' \in AL_2(D) \tag{5}$$

defines an operator $\mathcal{F}_D : AL_2(D) \to AL_2(D)$, which is unitary equivalent to $\mathcal{F}$. The function

$$g : D \to \Omega \tag{6}$$

denotes here the inverse of the Riemann mapping. The matrix representation comes into scope if we take infinite sequences composed of the Taylor coefficients of the functions in $AL_2(D)$. The operator $\mathcal{F}_D$ is represented then by an infinite matrix $\mathcal{M}$ acting on an appropriate defined space of sequences.

Corollary 2.3 *The Friedrichs operator $\mathcal{F}$ and Schur complement operator $\mathcal{S}$ of the simply connected domain $\Omega = g(D)$ with appropriately smooth conformal mapping g are, respectively, unitary equivalent to the conjugate linear and complex linear operators*

$$\mathcal{MC} \text{ and } \frac{1}{2}\left(\mathcal{I} - \mathcal{CMC}\right),$$

which are defined by the infinite matrix $\mathcal{M}$. □

Remark 2.4 To be more precise $\mathcal{M}$ acts on the space of complex sequences

$$\ell_{(2,-1)} := \left\{ f \mid f = (f_0, f_1, f_2, \dots)^T \text{ fulfilling } \sum_{n=0}^{\infty} \frac{|f_n|^2}{n+1} < \infty \right\}.$$

The entries of $\mathcal{M}$ are defined by series composed of the coefficitents of the Taylor series expansions of the conformal mapping g and of the reciprocal of its derivative. There are domains given by the corresponding function g for which the series defining the entries of $\mathcal{M}$ are divergent. We formulate conditions on the conformal mapping g assuring the existence of the matrix representation. $\square$

The investigation of $\mathcal{F}_D$ or of the structure of its matrix representant $\mathcal{M}$ reveals several properties of the studied operators in dependence on the problem domain.

Theorem 2.5 *Let η be the conformal map of the domain Ω onto $\tilde{\Omega}$, which are both domains satisfying the conditions of Theorem 2.1. We then have*

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \sup_{w\in\Omega} |\sin(\arg \eta'(w))|, \tag{7}$$

$$\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \sup_{w\in\Omega} |\sin(\arg \eta'(w))|. \tag{8}$$

$\square$

Remark 2.6 The continuity results in Theorem 2.5 can also be formulated in geometric terms involving the boundary of the domain Ω:

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \max_{w\in\partial\Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|, \tag{9}$$

$$\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \max_{w\in\partial\Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|, \tag{10}$$

where $\tau(w)$ denotes the unit tangent vector to $\partial\Omega$, being the conformal image of the unit circle, at the point $w = g(z)$. $\square$

Lemma 2.7 *Let be $\rho \geq 1$ integer. $\mathcal{M} \in \mathbb{C}^{\rho\times\rho}$ if and only if* (6) *is a polynomial of order ρ. In this case $\mathcal{M} \in \mathbb{C}^{\rho\times\rho}$ is of rank ρ.* $\square$

Lemma 2.8 *The matrix $\mathcal{M}$ and the correponding operators $\mathcal{F}$ and $\mathcal{S}$ are of finite rank if and only if* (6) *is a fractional rational transformation.* $\square$

The special class of domains described in Lemma 2.7 are called (classical) quadrature domains and there is extensive research on such domains. The result of Lemma 2.8 is already known for arbitrary quadrature domains, see e.g. [39], where it is proved without the use of conformal mapping. Its use in the proof, however, reveals not only the dimension of the kernel for simply connected domains but also its structure.

Theorem 2.9 *Let Ω be a simply connected quadrature domain of order $\rho+1$, $\rho \geq 0$. Then the kernel of the associated operators $\mathcal{F}$ and $\mathcal{I}-2\mathcal{S}$ has codimension $\rho+1$ and is spanned with the help of the denominator of the Schwarz function of the domain.* □

Lemma 2.10 *Let $\Omega = g(D)$ have finite area. Then at least one of the entries of $\mathcal{M}$ is not zero. There are domains of infinite area of which corresponding matrix $\mathcal{M}$ is identically the zero matrix.* □

After the investigation of the operators we study their spectra. We particularly focus on the inf-sup constant $\beta_0(\Omega)$, the square of which is the least positive eigenvalue of the Schur complement operator. We look for possibilities how its eigenvalues can be determined or estimated utilizing the properties of the conformal mapping. Our starting point is an alternate form of the Friedrichs inequality [21]: there exists a constant $\Gamma_\Omega \geq 1$ depending only on the shape of the domain Ω such that for all conjugate harmonic functions u and v in $L_2(\Omega)$ there follows

$$\int_\Omega u^2 dA \leq \Gamma_\Omega \int_\Omega v^2 dA \text{ provided } \int_\Omega u dA = 0. \tag{11}$$

We obtain several results on this basis:

Corollary 2.11 *Let $\Omega = g(D)$ be such that the derivative g' of the conformal map is continuous on the closure of D and $|g'|$ has a positive lower and upper bound on ∂D. Then we have*

$$\frac{1}{\sqrt{2}} \cdot \frac{\inf_{z\in\partial D} |g'(z)|}{\sup_{z\in\partial D} |g'(z)|} \leq \beta_0(\Omega) \leq \frac{1}{\sqrt{2}} \tag{12}$$

for the inf-sup constant of the domain Ω. The equalities hold for $g(z) = z$, that is, $\Omega = D$. □

Corollary 2.12 *Let the domains Ω and $\tilde{\Omega}$ have smooth boundaries such that the derivative of the bijective conformal mapping η of Ω onto $\tilde{\Omega}$ is continuous on the closure of Ω and $|\eta'|$ has a positive lower and upper bound on $\partial\Omega$. Then there follows*

$$\frac{\inf_{\partial\Omega} |\eta'|}{\sup_{\partial\Omega} |\eta'|} \beta_0(\Omega) \leq \beta_0(\tilde{\Omega}).$$

□

If η is near to the identity mapping in the sense that the maximum norm of $\eta' - 1$ is small, then we obtain

Theorem 2.13 *If the bijective conformal mapping η of Ω onto $\tilde{\Omega}$ is such that $|\eta'(w) - 1| \leq \varepsilon < 1$ for all $w \in \overline{\Omega}$, then there follows*

$$\frac{1-\varepsilon}{1+\varepsilon}\beta_0(\Omega) \leq \beta_0(\tilde{\Omega}) \leq \frac{1+\varepsilon}{1-\varepsilon}\beta_0(\Omega) \tag{13}$$

for the inf-sup constants of the domains. □

Remark 2.14 Let the domain Ω have $C^{1,\alpha}$ smooth boundary for some $0 < \alpha < 1$. Let the sequence of domains Ω_n, $n = 1, 2, \ldots$, tend to the domain Ω in the sense that for their corresponding conformal mappings one has the following: for every $0 < \varepsilon < 1$ there exists a natural number N such that

$$\sup_D |g_n' - g'| < \varepsilon, \tag{14}$$

whenever $n > N$. Then by Theorem 2.13 there follows

$$\lim_{n\to\infty} \beta_0(\Omega_n) = \beta_0(\Omega),$$

i.e. one has the convergence of the inf-sup constants. Therefore we have obtained a sufficient condition for the convergence of the inf-sup constants of a convergent domain sequence to the inf-sup constant of the limit domain. If one has a condition weaker than (14), then the convergence of the inf-sup constants can not be guaranteed. The example $g_m(z) = z - \frac{c}{m}z^m$, where $m > 1$ is an integer and $0 < |c| \leq 1$ illustrates this phenomenon. If $\Omega_m = g_m(D)$, then the domain sequence (Ω_m) tends to the unit disc but

$$\lim_{m\to\infty} \beta_0^2(\Omega_m) = \frac{1}{2} - \frac{|c|}{4} < \frac{1}{2} = \beta_0^2(D).$$

We have computed this example with the help of the matrix representation derived for the studied operators. □

The multiplicity of the eigenvalues of the Schur complement operator can also be examined with the help of conformal mapping.

Theorem 2.15 *Let the bijective conformal mapping of the unit disc be of the form*

$$g(z) = \sum_{n=0}^{\infty} a_{nM+1} z^{nM+1}, \tag{15}$$

where $a_1 \neq 0$ and $M \geq 2$ is an arbitrary integer, then there are eigenvalues of the Schur complement operator with multiplicity more than 1. □

Remark 2.16 Domains involved in Theorem 2.15 have an M-fold rotational symmetry. We also give an example for a domain such that the square of its inf-sup constant is a double eigenvalue of the Schur complement operator. □

Remark 2.17 With the help of the matrix representation of the Schur complement operator and of Corollary 2.11 we can estimate the inf-sup constants of some domains (star-shaped domains, convex domains or nearly circular domains for example). We calculate also some examples for domains of which boundary has corners: the wedge, the annular sector, etc. □

3 Representation results

This section contains results about representation formulae for Stokes functions based mainly on [55]. In this section we omit complex formalism because we also consider three-dimensional domains, but the connection of this section to the previous one is explained.

First we prove the equivalence of two known results from [29] and [30]. This proof is based on a similar representation of conjugate pairs, which are harmonic functions $\vec{q}$ and p on a spatial domain Ω connected by

$$\operatorname{rot}\vec{q}(\vec{x}) = -\nabla p(\vec{x}), \quad \operatorname{div}\vec{q}(\vec{x}) = 0 \quad \text{for} \quad \vec{x} \in \Omega. \tag{16}$$

Lemma 3.1 *Assume $\Omega \subset \mathbb{R}^3$ is a star-shaped domain with respect to the origin. Then the pair $(\vec{q}, p)$ on Ω is a conjugate pair iff there exist scalar harmonic functions φ and ϕ in Ω such that*

$$\vec{q}(\vec{x}) = \nabla\varphi(\vec{x}) + \vec{x} \times \nabla\phi(\vec{x}), \tag{17}$$

$$p(\vec{x}) = \phi(\vec{x}) + \vec{x} \cdot \nabla\phi(\vec{x}), \tag{18}$$

for $\vec{x} \in \Omega$. These harmonic functions are uniquely determined by the pair $(\vec{q}, p)$ under the normalization $\varphi(0) = \phi(0) = p(0) = 0$, and they are given by the formulae

$$\phi(\vec{x}) = \int_0^1 \frac{1}{\tau^2} e^{1-\frac{1}{\tau}} p(\tau\vec{x}) d\tau, \tag{19}$$

$$\varphi(\vec{x}) = \int_0^1 \vec{x} \cdot \vec{q}(\tau\vec{x}) d\tau \left(= \int_0^1 \vec{x} \cdot (\vec{q}(\tau\vec{x}) - \tau\vec{x} \times \nabla\phi(\tau\vec{x}))\, d\tau \right). \tag{20}$$

□

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^3$ be a star-shaped domain with respect to the origin. The representation formulae given in [29] and [30] are equivalent.* □

Remark 3.3 We further connect the results contained in Theorem 3.2 to a third representation – called Papkovich-Neuber representation – given in [38]. □

We also study the question how to avoid the restriction of star-shapedness for the domain in Theorem 3.2.

Theorem 3.4 *Let $\Omega \subseteq \mathbb{R}^3$ be a domain. $(\vec{v}, p)$ is a Stokes pair on Ω iff there are harmonic functions $\vec{w}$ and ψ on Ω such that*

$$\vec{v} = -\frac{1}{2}\nabla(\vec{x}\cdot\vec{w}) - \frac{1}{2}\mathit{rot}(\vec{x}\times\vec{w}+\psi\vec{x}) + \vec{w}, \quad (21)$$

$$p = -2\,\mathit{div}\,\vec{w}. \quad (22)$$

Moreover, these harmonic functions are

$$\vec{w} = \frac{2}{3}\vec{v} - \frac{1}{6}(p\vec{x} - \vec{x}\times \mathit{rot}\,\vec{v}), \quad (23)$$

$$\psi = -\frac{1}{6}\vec{x}\cdot \mathit{rot}\,\vec{v}. \quad (24)$$

Next we examine similar representations for the solutions Navier's equation for the linear elasticity problem. (This can be seen as another generalization of the formulae given in [29] and [30].)

Theorem 3.5 *Let $\Omega \subset \mathbb{R}^3$ be a star-shaped domain with respect to the origin, and set $\nu \in \mathbb{R}$, $\nu \neq \frac{3}{4}, 1$. The functions $\vec{v} \in C_2(\Omega)$ and $p \in C_1(\Omega)$ satisfy*

$$\Delta\vec{v} = \nabla p, \quad \mathit{div}\,\vec{v} = \nu p \quad (25)$$

if and only if there exists a harmonic function $\vec{h} \in C_2(\Omega)$ such that

$$\vec{v}(\vec{x}) = -\frac{1}{2}\left(\vec{x}\,\mathit{div}\,\vec{h}(\vec{x}) + \vec{x}\times\mathit{rot}\,\vec{h}(\vec{x})\right) + \left(\frac{3}{2}-2\nu\right)\vec{h}(\vec{x}), \quad (26)$$

$$p(\vec{x}) = -2\,\mathit{div}\,\vec{h}(\vec{x}), \text{ for } \vec{x}\in\Omega. \quad (27)$$

The harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}(\vec{x}) = \frac{2}{3-4\nu}\left(\vec{v}(\vec{x}) - \frac{1}{4}\left(p(\vec{x})\vec{x} - \frac{1}{1-\nu}\vec{x}\times\mathit{rot}\,\vec{v}(\vec{x})\right) + \vec{x}\times\nabla\phi(\vec{x})\right), \quad (28)$$

where the function ϕ is harmonic in Ω and defined by

$$\phi(\vec{x}) = -\frac{1}{4(1-\nu)}\int_0^1 t^{4(1-\nu)}\vec{x}\cdot\mathit{rot}\,\vec{v}(t\vec{x})dt. \quad (29)$$

□

To complete the generalization of the representation theorems we consider also the two-dimensional counterpart of Theorem 3.5.

Theorem 3.6 *Let $\Omega \subseteq \mathbb{R}^2$ be a domain and set $\nu \in \mathbb{R}$, $\nu \neq \frac{1}{2}, 1$. The functions $\vec{v} \in C_2(\Omega)$ and $p \in C_1(\Omega)$ satisfy* (25) *iff there exists a harmonic function $\vec{h} \in C_2(\Omega)$ such that*

$$\vec{v}\,(\vec{x}) = -\frac{1}{2}\left(\vec{x}\,\mathrm{div}\,\vec{h}\,(\vec{x}) + \vec{x}^{\perp}\,\mathrm{rot}\,\vec{h}\,(\vec{x})\right) + (1-2\nu)\vec{h}\,(\vec{x}) \qquad (30)$$

$$p\,(\vec{x}) = -2\,\mathrm{div}\,\vec{h}\,(\vec{x}) \quad \text{for } \vec{x} \in \Omega; \qquad (31)$$

this harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}\,(\vec{x}) = \frac{1}{1-2\nu}\left(\vec{v}\,(\vec{x}) - \frac{1}{4}\left(p\,(\vec{x})\,\vec{x} - \frac{1}{1-\nu}\vec{x}^{\perp}\,\mathrm{rot}\,\vec{v}\,(\vec{x})\right)\right) \quad \text{for } \vec{x} \in \Omega. \qquad (32)$$

□

Remark 3.7 The formulae (30) and (31) can be put in complex form equivalent to the formulae used in the previous Section 2. □

We end this section by formulating two connections between the two-dimensional and three-dimensional representation formulae.

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