Instability of Risk Measures

Summary of Ph.D. thesis

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Introduction

Measuring and minimizing risk under various conditions are perhaps the most fundamental and most difficult problems in financial theory. As an indication of the complexity of the issue, the various participants of finance (academics, bankers and regulators) are still in disagreement on the definition of financial risk and, accordingly, the number of risk concepts and risk measures introduced in the literature and in practice is very large. Moreover, in spite of the repeated warnings of the academic world, financial practitioners and regulators have been almost exclusively relying on ad-hoc risk measurement and management techniques, with a very weak theoretical foundation. This has led to an unfortunate divergence of the evolution of theory from that of practice: on the one hand, practitioners have become reluctant to acknowledge new theoretical results, on the other hand, the mathematical theory of risk has become insensitive to practical issues.

One example of the neglected issues is the performance evaluation of these methods in the realistic situation when the background processes, or at least their parameters are not known, therefore investment risk must be estimated on the basis of empirical data. The literature of financial risk generally gives priority to mathematical elegance and abstract theoretical reasoning, while (usually implicitly) assuming that the background processes are known. In this approach the question of estimation error is regarded as a secondary issue, which must be preceded by the abstract theoretical foundation of the problem.

In the dissertation we use several examples to demonstrate that this approach is fundamentally wrong. We study the sensitivity of risk measures with respect to sampling noise through the portfolio selection problem, which is one of the basic tasks of financial mathematics. Based on the literature and our own research findings we study the noise sensitivity of a number of different risk measures and we are led to the conclusion that some of the risk measures with great theoretical and/or practical importance – the so called Value-at-Risk, and the coherent risk measures – are strongly and inherently unstable. Using simple mathematical considerations, we generalize these results to a very wide range of risk measures, thereby raising a question about their practical applicability. All this evidently indicates that the problem of sensitivity with respect to sampling noise is of paramount importance for the theoretical foundation of risk measurement.

To carry out these studies, – besides simulations and elementary mathematics – we relied heavily on the replica method borrowed from the theory of disordered systems, thus providing a few further examples for the application of this technique beyond physics. At the end of the thesis we briefly describe how the methods used in the dissertation to characterize sampling noise could be applied in a much more general framework, to examine the noise sensitivity of models introduced in various sciences and disciplines.

Methodology

To characterize the noise sensitivity of portfolio optimization under different risk measures and asset return distributions, we use a quantity denoted by q_0, which is defined as the ratio of the risk of the sample based estimation of the optimal portfolio, and the risk in the real optimum. One of the advantages of q_0 is that it is dimensionless, therefore it can be used to compare the noise sensitivity of very different models, moreover it provides a measure of the estimation error which is easy to interpret for investors. Besides determining the value of q_0 we also investigated the conditions for, and the probability of, the existence of the optimum.

We studied the behaviour of q_0 and the existence of the optimum using the replica method, as well as numerical simulations. This is possible due to the fact that portfolio optimization based on a finite sample is equivalent to a statistical physics problem, in which portfolio weights correspond to the generalized coordinates (e.g. position or spins) of the system, while couplings can be derived from the finite sample. Since the sample, and therefore the couplings are random variables, the portfolio problem is analogous to certain models of spin glasses and disordered systems (e.g. the Sherrington-Kirkpatrick model), so the replica method applied as an analytical tool to examine such systems can be used effectively also in a financial context.

Where the replica method could not cope with the complexity of the model, or we simply wanted to check our replica-based analytical results, we applied Monte Carlo simulations . In other words, we used a predefined distribution to generate the samples, and for each sample, we determined the optimal portfolio. This allowed us to keep track of q_0 and the existence of the optimum from sample to sample, and to estimate the mean of q_0 and the probability of the existence of the optimum.

At the end of the dissertation we use simple mathematical considerations to generalize our special results on different risk measures and derive a set of very general statements.

New results

1. Using the replica method, we reproduced the $q_0 = (1 - N/T)^{-1/2}$ formula, which describes the dependence of the estimation error of the classical Markowitz problem on the market size N and the sample size T. Although this result is well known, it has not been derived via the replica method in the literature so far. As the classical Markowitz problem is basically the simplest of the portfolio selection problems, its derivation via the replica method is a good starting point for the analysis of more complicated models (e.g. downside risk measures).

2. We employed the replica method to examine the noise sensitivity of the minimization of some important downside risk measures (Value-at-Risk, Expected Shortfall, semivariance) in the case when risk is estimated parametrically (rather than historically). (In other words, we make an assumption concerning the form of the asset return distribution, and only estimate its parameters.) We conclude that, similarly to the historical estimation of Expected Shortfall, the parametric estimation of downside risk measures is also unstable, and the optimum does not exist with finite probability. In the 'thermodynamic' limit (when both the number of assets N and the sample size T go to infinity, while the ratio $r = N/T$ is fixed) the parameter plane spanned by the relative sample size r and the confidence level α of Value-at-Risk or Expected Shortfall is divided into two phases along a well defined critical curve: below the curve the optimum exists, while above the curve it does not. We also demonstrate that approaching the phase boundary from the feasible region the measurement error diverges according to $q_0 \sim (r_c - r)^{-1/2}$, where r_c is the critical value of N/T. Therefore, the problem under examination seems to belong to the same universality class as the other portfolio selection problems studied in the literature and in the dissertation. A further important result is that even though (in agreement with expectations) the parametric estimation is somewhat more stable then the historical one, it still does not resolve the main issue, namely that the the optimization problem is unfeasible with finite probability.

3. As the assumption of independent and identically distributed price fluctuations is oversimplifying, we also studied the noise sensitivity of portfolio estimation under the somewhat more realistic GARCH models (more specifically the so called CCC-GARCH(1,1) model). Based on numerical methods, we concluded that as long as the investor is aware of the presence of the GARCH effect (that is, he knows the character of the process, but does not know its parameters), the estimation error of optimization simply behaves according to the well known formula: $q_0 \sim (1 - N/T)^{-1/2}$. If, however, in spite of the presence of the GARCH effect the investor naively assumes independent, Gaussian fluctuations, then the error q_0 will be much larger, and it does not converge to one even when $T \to \infty$. Therefore, these results serve as a good demonstration of the potential size of model specification error.

4. Via elementary mathematics, we demonstrated that the instability related to the estimated optimum is not a specific property of the studied risk measures, but originates from some of their fundamental properties. Thus, it turned out that the majority of the risk measures applied in theory and practice are unstable (including Value-at-Risk, and the coherent risk measures, which are increasingly popular among financial mathematicians), which makes them inappropriate for practical use. Our reasoning can also provide a starting point for the development of new, more stable techniques of risk measurement.

5. Furthermore, we recognized that the applicability of the methods applied throughout the dissertation goes far beyond the scope of portfolio optimization; these methods can also be used to study the sampling noise sensitivity of models in any discipline, so that we can provide a lower estimate for the expected estimation error, as well as for the acceptable sample size. Moreover, this connection between the theory of portfolio optimization and that of statistical estimation makes it possible to effectively use the methods of statistical physics in areas where so far traditional mathematical methods prevailed.

Conclusions

- From a financial point of view, our most important conclusion is that the theoretical foundation of risk measures must take into account the practical aspects of risk measurement and minimization, such as estimation from a finite sample. Neither the risk measures used in practice nor those proposed by theoreticians satisfy this criterion, therefore the problem of risk measurement calls for a radically different approach.

- In addition, we extended the range of non-physical (more precisely financial) applications of the replica method, and at the end of the dissertation we showed how we can go beyond the problem of portfolio optimization and apply this methodology to the area of mathematical statistics.

Publications

- I. Varga-Haszonits and I. Kondor, *Noise Sensitivity of Portfolio Selection in Constant Conditional Correlation GARCH Models*, Physica A 385, p307-318 (2007)
- I. Varga-Haszonits and I. Kondor, *The Instability of Downside Risk Measures*, J. Stat. Mech. 12, P12007 (2008)
- I. Kondor and I. Varga-Haszonits, *Feasibility of Portfolio Optimization under Coherent Risk Measures*, arXiv:0803.2283v3 [physics.soc-ph], submitted to Quantitative Finance (2008)
- I. Kondor and I. Varga-Haszonits, *Divergent estimation error in portfolio optimization and in linear regression*, Eur. Phys. J. B 64, p601-605 (2008)