Rigidity matroids and inductive constructions of graphs and hypergraphs

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Chapter 1

Introduction

Rigidity theory investigates the rigidity and flexibility of frameworks. One well-studied example is the following: designated points of the d-dimensional Euclidean space represent universal joints that are connected with fixed-length bars. These structures are called *bar-and-joint frameworks*. A bar-and-joint framework is *flexible* if there is a continuous motion that changes the distance between at least one pair of joints not connected by a bar without changing the lengths of the bars. Otherwise the framework is *rigid*. The problem of rigidity of bar-and-joint frameworks is central in rigidity theory, however, this field of mathematics studies many different types of frameworks. For instance instead of the bars we can put cables and struts between the joints or the joints can be replaced by higher dimensional bodies. Instead of the Euclidean space the framework can live on a surface of the space or in an affine or projective space. There are also frameworks whose underlying structure is a hypergraph and point sets defined by hyperedges have to preserve for instance some affine relation or the volume of the polyhedron they define.

A number of questions in rigidity theory were motivated by applications, such as engineering, biology, chemistry, robotics. Most of the topics discussed in this work focus on the rigidity of bar-and-joint frameworks but we will also consider frameworks on hypergraphs in Chapters 5 and 7.

The thesis is organized as follows. In the remainder of this chapter we review the standard theory of rigid structures and sparse graphs. We also give a brief introduction of symmetric rigidity any scene analysis.

Chapter 2 focuses on the two-dimensional generic rigidity matroid, $\mathcal{R}_2(G)$. We investigate the correspondence between the high connectivity of $\mathcal{R}_2(G)$ and the uniqueness of the graph defining this matroid. The main result of this chapter is that if G is 7-connected then G is uniquely determined by $\mathcal{R}_2(G)$. This implies that if $\mathcal{R}_2(G)$ is 11-connected, then it uniquely determines G.

In Chapter 3 we give upper and lower bounds for the edge number of minimally highly vertex-redundantly rigid graphs in $\mathbb{R}^d$ for every d.

The main result of Chapter 4 is a characterization for the existence of a two-dimensional infinitesimally rigid realization with two coincident vertices.

In Chapter 6 we consider symmetric frameworks in the plane. We characterize minimally symmetry-forced rigid graphs for point groups $\mathcal{C}_k$ for $k \geq 1$ and for $\mathcal{D}_k$ where k is odd.

In Chapter 5 we first give a constructive characterization of 4-uniform (1,3)-tight hypergraphs. Then using this result we give a combinatorial characterization of generically projectively rigid hypergraphs on the projective line. With the construction we can also characterize 4-uniform affinely rigid hypergraphs in the plane.

In Chapter 7 we focus on the symmetric version of scene analysis. The main result of the chapter is a $\mathcal{C}_3$-symmetric version of the constructive characterization given in Chapter 5. Using this construction we characterize $\mathcal{C}_3$-symmetric two-dimensional 'symmetry-generic' minimally flat scenes.

1.1 Rigidity and infinitesimal rigidity of bar-and-joint frameworks

A d-dimensional *bar-and-joint framework* or *framework* (G, p) is a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. We say that the framework (G, p) is a *realization* of the graph G in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are said to be *equivalent* if $\| p(u) - p(v) \| = \| q(u) - q(v) \|$ holds for every edge $uv \in E$. If $\| p(u) - p(v) \| = \| q(u) - q(v) \|$ for every pair $u, v \in V$, then (G, p) and (G, q) are *congruent*. We say that a framework (G, p) is *rigid* if there exists an $\varepsilon > 0$ such that if (G, q) is equivalent to (G, p) and $\| p(v) - q(v) \| < \varepsilon$ for all $v \in V$ then (G, q) is congruent to (G, p).

An other possible equivalent definition of rigidity is described below. A *flexing* of the framework (G, p) is a continuous function $\pi : (-1, 1) \times V \to \mathbb{R}^d$ such that $\pi(0) = p$, and the frameworks (G, p) and $(G, \pi(t))$ are equivalent for all $t \in (-1, 1)$. The flexing π is *trivial* if the frameworks (G, p) and $(G, \pi(t))$ are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial flexing otherwise it is called *rigid*. (G, p) is *minimally rigid*, if it is rigid but after deleting any edge of G the framework becomes flexible.

It is not difficult to see that framework (G, p) is rigid in $\mathbb{R}^1$ if and only if G is connected. Deciding rigidity of frameworks in higher dimensions is a hard problem, Abbot [1] showed that this is NP-hard for even two-dimensional frameworks.

To make these problems more tractable we often consider 'generic' realizations. A framework (G, p) is *generic* if the set of coordinates of the points $p(v)$, $v \in V$, is algebraically independent over the rationals. Graph G is said to be *(generically) rigid* if it has a rigid generic realization.

An *infinitesimal motion* is an assignment of *infinitesimal velocities* to the vertices of G, $m : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(m(u) - m(v)) = 0 \text{ for every pair } u, v \text{ with } uv \in E.$$

For instance if π is a smooth flexing, then with the definition $\dot{\pi}(v) = \frac{d}{dt}\pi(t, v)|_{t=0}$, $\dot{\pi}$ is an infinitesimal motion. An infinitesimal motion is *trivial* if it is an infinitesimal motion of $(K_{|V|}, p)$. A framework is *infinitesimally flexible*, if it has a non-trivial infinitesimal motion, otherwise it is *infinitesimally rigid*. Gluck [15] proved that if a framework (G, p) is infinitesimally rigid, then it is rigid. The converse does not hold in general but, for generic frameworks rigidity and infinitesimal rigidity are equivalent [3, 4].

The set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{d|V|}$ given by the system of $|E|$ linear equations. If we collect these linear equations into a matrix we get the d-dimensional *rigidity matrix* of the framework. This is the matrix $R_d(G, p)$ of size $|E| \times d|V|$, where, for each edge $uv \in E$, in the row corresponding to uv, the entries in the d columns corresponding to the vertices u and v contain the d coordinates of $(p(u) - p(v))$ and $(p(v) - p(u))$, respectively, and the remaining entries are zeros.

Thus $m \in \mathbb{R}^{d|V|}$ is an infinitesimal motion if and only if $R(G, p)m = 0$. Each isometry of $\mathbb{R}^d$ gives rise to a smooth motion of (G, p) and hence to a trivial infinitesimal motion of (G, p). Hence

Lemma 1.1.1 *[68] Let (G, p) be a framework in $\mathbb{R}^d$. Then*

$$\text{rank} R(G, p) \leq S(n, d)$$

where $n = |V|$ and

$$S(n, d) = \begin{cases} nd - \binom{d+1}{2}, & \text{if } n \geq d + 2; \\ \binom{n}{2}, & \text{if } n \leq d + 1. \end{cases}$$

Thus a framework (G, p) is infinitesimally rigid in $\mathbb{R}^d$ if and only if $\text{rank} R(G, p) = S(n, d)$. We say that (G, p) is *independent* if the rows of $R(G, p)$ are linearly independent. An independent and infinitesimally rigid framework is called *minimally infinitesimally rigid.*

$R_d(G,p)$ defines the d-dimensional rigidity matroid of (G,p) on the ground set E by linear independence. Since the entries of the rigidity matrix are polynomial functions with integer coefficients, any two generic d-dimensional frameworks (G,p) and (G,q) have the same rigidity matroid. We call this the d-dimensional rigidity matroid $\mathcal{R}_d(G)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. It follows from the discussions above that a graph G on n vertices is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = S(n,d)$. $\mathcal{R}_d(G)$ is characterized for $d = 1, 2$, see Section 1.2 for details. It remains an open problem to find good characterizations for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

1.2 Sparse graphs and count matroids

Theorem 1.2.1 *[34] For integers $k \geq 0$ and l, the following definition gives the independent sets of a matroid $\mathcal{M}_{k,l}(H)$ on the edges of a hypergraph $H = (V,F)$: A set of hyperedges F is independent if and only if all non-empty subsets F' on vertices V' satisfy $|F'| \leq k|V'| - l$.*

The matroids defined in Theorem 1.2.1 are called *count matroids* or sometimes *sparsity matroids*. A hypergraph $H = (V,F)$ is *independent* in $\mathcal{M}_{k,l}(H)$ if F is independent in $H = (V,F)$. Hypergraphs that are independent in $\mathcal{M}_{k,l}(H)$ are called (k,l)-*sparse*. If they also satisfy $|E| = k|V| - l$ then are called (k,l)-*tight* hypergraphs.

If $l \geq 2k$, the matroid will have all graph edges dependent. Therefore, for graphs we should assume that $l < 2k$. We also need $l < k$ to make loops into independent elements. In general, we need $l < sk$ to make the matroid non-trivial on s-tuple edges.

As mentioned earlier a framework (G,p) is rigid in one dimension if and only if G is connected. Therefore minimally rigid graphs in one dimension are trees. Thus $\mathcal{R}_1(G)$, the cycle matroid of G and the $\mathcal{M}_{1,1}(G)$ count matroid are isomorphic.

$S(n,2) = 2n - 3$ hence if G is independent in $\mathcal{R}_2(G)$ then it has to be $(2,3)$-sparse. A celebrated result of Laman [32] (see Theorem 1.3.1) states that $\mathcal{R}_2(G)$ is isomorphic to $\mathcal{M}_{2,3}(G)$.

There are efficient algorithms for testing independence in count matroids, see [7]. Thus independence in $\mathcal{R}_1(G)$ and $\mathcal{R}_2(G)$ can also be tested efficiently.

Count matroids and sparse graphs are frequently used in rigidity theory. When one tries to find a characterization for some rigidity matroid then often gets a necessary sparsity condition for independent graphs. Sometimes it turns out that the spar-

sity condition is sufficient (e.g. for $\mathcal{R}_1(G)$ and $\mathcal{R}_2(G)$). For higher dimensions we can also obtain necessary sparsity conditions for independence in $\mathcal{R}_d(G)$, however these conditions are not sufficient. For instance if $d = 3$ and $n \geq 3$ then $S(n, d) = 3n - 6$. Hence every edge set with at least two edges has to be $(3, 6)$-sparse.

1.3 Constructive characterizations

By constructive characterization or inductive construction we mean the following approach for describing a certain class of graphs, $\mathcal{G}$. A set of operations is given so that performing any of the given operations on a graph in $\mathcal{G}$ gives another graph in $\mathcal{G}$, and moreover, we can construct every graph in $\mathcal{G}$ by a sequence of these operations starting from a small initial set of basic instances in $\mathcal{G}$.

Constructive characterizations are often used in rigidity theory as they provide a useful tool for proving rigidity or independence of certain families of frameworks.

The following theorem summarizes the results of Henneberg, Laman [32] and Tay and Whiteley [58].

Theorem 1.3.1 *For a graph G, the following are equivalent:*

1. *G is minimally generically rigid in the plane;*

2. *G is $(2, 3)$-tight;*

3. *G can be built up from a single edge by a sequence of the following operations:*
   (i) add a new vertex z and connect it to two different existing vertices x and y,
   (ii) subdivide an edge uv with a new vertex z and add a new edge between z and an existing vertex w different from u and v.

Operations (i) and (ii) are called the two-dimensional (Henneberg-)0-extension and (Henneberg-)1-extension, respectively. We prove a result in Chapter 4 similar to Theorem 1.3.1 in order to characterize graphs that have a rigid realization in which two designated vertices coincide.

Let $K(i, j, k)$ denote the following operation: pinch j edges of the graph with a new vertex z, add i loops incident to z and k edges between z and old vertices. Let P_l denote the graph with a single vertex and l loops. With this notation the two-dimensional 0-extension is $K(0, 0, 2)$ while the 1-extension is $K(0, 1, 2)$. These two operations have d-dimensional variants for every $d \geq 2$. The d-dimensional 0-extension is $K(0, 0, d)$ and the d-dimensional 1-extension is $K(0, 1, d)$. As they are

known to preserve both d-dimensional rigidity and independence we will use the d-dimensional extensions in the proofs of Chapter 3.

Theorem 1.3.2 *[13] For $1 \leq l \leq k$ a graph is (k,l)-tight if and only if it can be obtained from P_{k-l} by iteratively applying $K(i,j,k)$ operations with $i+j \leq k-1$, $i,j \geq 0$, $i \leq k-l$.*

The graph is $(k,0)$-tight if and only if it can be built up from P_k by iteratively applying $K(i,j,k)$ operations with $i+j \leq k$, $i,j \geq 0$

We prove a version of Theorem 1.3.2 for 'gain graphs' (that are directed graphs whose edges are labeled with the elements of a group) for $k=2$, $l=0$ and $l=1$ in Chapter 6 which provides a constructive characterization for two classes of gain-graphs. The construction then is used to characterize symmetrically rigid graphs.

1.4 Rigidity with symmetry

1.4.1 Symmetric realizations

We shall first need the definition of symmetric graphs and hypergraphs. As graphs are special hypergraphs we give the definitions for hypergraphs only. Let $H=(V,F)$ be a hypergraph. An *automorphism* of H is a permutation $\pi : V \to V$ such that $\{v_1,\ldots,v_k\} \in F$ if and only if $\{\pi(v_1),\ldots,\pi(v_k)\} \in F$. The set of all automorphisms of H forms a group under decomposition known as the *automorphism group* $\mathrm{Aut}(H)$ of H.

Let $\mathcal{S}$ be a group. An *action* of $\mathcal{S}$ on H is a group homomorphism $\rho : \mathcal{S} \to \mathrm{Aut}(H)$. An action ρ is called *free* if $\rho(g)(v) \neq v$ for any $v \in V$ and any non-identity $g \in \mathcal{S}$. We say that a hypergraph H is $(\mathcal{S},\rho)$-*symmetric* if $\mathcal{S}$ acts on H by ρ. If ρ is clear from the context, we will simply denote $\rho(g)(v)$ by $g \cdot v$ or gv. (S is often a subgroup of $\mathrm{Aut}(H)$ and in such a case ρ is usually an embedding.) Note that, for $g \in \mathcal{S}$, $\{v_1,\ldots,v_k\} \in F$ if and only if $\{gv_1,\ldots,gv_k\} \in F$, and hence there is an induced action of $\mathcal{S}$ on F defined by $g \cdot \{v_1,\ldots,v_k\} = \{gv_1,\ldots,gv_k\}$.

A *discrete point group* (or simply a *point group*) is a finite discrete subgroup of $\mathcal{O}(\mathbb{R}^d)$, the *orthogonal group* of dimension d, i.e., the set of $d \times d$ orthogonal matrices over $\mathbb{R}$. In this work we only consider two-dimensional point groups. These are classified into two classes, *groups $\mathcal{C}_k$ of k-fold rotations* and *dihedral groups $\mathcal{D}_k$* of order $2k$. For a special case, $\mathcal{D}_1$ consists of a mirror-reflection and the identity.

Suppose that H is $(\mathcal{S},\rho)$-symmetric for a point group $\mathcal{S}$. A joint-configuration p is said to be $(\mathcal{S},\rho)$-*symmetric* (or, simply, $\mathcal{S}$-symmetric) if

$$gp(v) = p(gv) \qquad \text{for all } g \in \mathcal{S} \text{ and for all } v \in V(H).$$

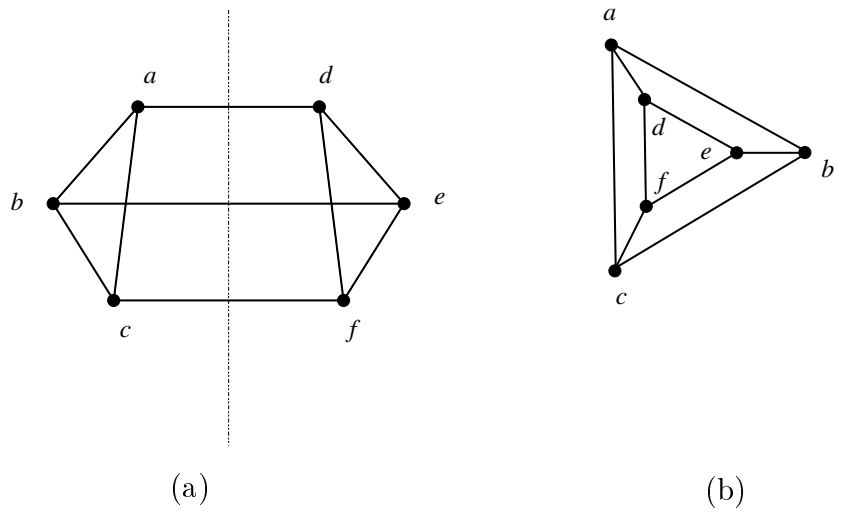

Figure 1.1: Two symmetric realizations of the triangular prism. Realization (a) is $\mathcal{D}_1$-symmetric and realization (b) is $\mathcal{C}_3$-symmetric.

Such a pair (H, p) is called an $(\mathcal{S}, \rho)$-*symmetric framework* (or simply an $\mathcal{S}$-symmetric framework or a symmetric framework).

1.4.2 Incidental versus forced symmetry of bar-and-joint frameworks

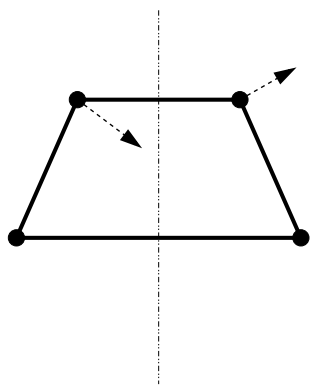

Figure 1.2: A reflection-symmetric framework which is symmetry-forced rigid but not rigid. The two arrows represent the velocities of a non-trivial and non-symmetric motion.

Symmetry is ubiquitous in both natural and man-made structures, hence studying the rigidity of symmetric frameworks has received much attention in the past several years. When investigating the rigidity of a symmetric bar-and-joint framework we can think of two different approaches. One of these is *incidental symmetry* in which one studies a symmetric framework that may move in unrestricted ways.

The other approach is *forced symmetry* where a framework must maintain symmetry with respect to a specific group throughout its motion.

We will consider symmetric frameworks in Sections 6 and 7. In Section 6 we characterize symmetry-forced rigid graphs for point groups $\mathcal{C}_k$ for every $k \geq 2$ and for $\mathcal{D}_k$ where k is even. In Section 7 we give a characterization for minimally flat $\mathcal{C}_3$-symmetric 2-scenes.

Notation

Graphs	
$G = (V, E)$	an undirected graph
$G[X]$	the subgraph of G induced by $X \subseteq V$
$E_G(X)$	the set of edges of $G[X]$
$i_G(X)$	$= \lvert E_G(X) \rvert$
$d_G(v)$	the degree of v in G for $v \in V$
$N_G(v)$	the set of neighbors of v in G
$V_G(F)$	the set of endvertices of edges in F for $F \subseteq E$
$d_F(v)$	$= d_{G[F]}(v)$ where $F \subseteq E$
$\Delta(G)$	the maximum degree in G
K_n	the complete graph with n vertices
C_n	the cycle on n vertices
Matroids	
$\mathcal{R}_d(G)$	the d-dimensional rigidity matroid of G
$r_d(G)$	the rank of G in the d-dimensional rigidity matroid
Groups	
$\mathcal{C}_k$	group of k-fold rotations
$\mathcal{D}_k$	dihedral group of order $2k$ generated by a k-fold rotation and a reflection

We may omit the subscripts referring to G (d) if the graph (dimension, respectively) is clear from the context.

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Chapter 2

Highly connected rigidity matroids have unique underlying graphs

Let $\mathcal{M}$ be a d-dimensional generic rigidity matroid of some graph G. In this chapter we consider the following problem, posed by Brigitte and Herman Servatius in 2006: is there a (smallest) integer k_d such that the underlying graph G of $\mathcal{M}$ is uniquely determined, provided that $\mathcal{M}$ is k_d-connected? Since the one-dimensional generic rigidity matroid of G is isomorphic to its cycle matroid, a celebrated result of Hassler Whitney implies that $k_1 = 3$. We extend this result by proving that $k_2 \leq 11$. To show this we prove that
(i) if G is 7-vertex-connected then it is uniquely determined by its two-dimensional rigidity matroid, and
(ii) if a two-dimensional rigidity matroid is $(2k-3)$-connected then its underlying graph is k-vertex-connected.

We also prove the reverse implication: if G is a k-connected graph for some $k \geq 6$ then its two-dimensional rigidity matroid is $(k-2)$-connected. Furthermore, we determine the connectivity of the d-dimensional rigidity matroid of the complete graph K_n, for all pairs of positive integers d, n.

Since no good characterization is known for independence in the three-dimensional rigidity matroid, the question whether k_3 exists seems more difficult. We note that three-dimensional versions of some of the key results that we used in the proofs exist as conjectures: Lovász and Yemini [35] conjecture that 12-vertex-connected graphs are rigid in three-space, while Jackson and Jordán [20] conjecture that if G is 5-vertex-connected and $\mathcal{R}_3(G)$ is 2-connected then G is redundantly rigid. The bounds on the vertex connectivity would be best possible in both conjectures.

2.1 Problem formulation and basic definitions

Let $\mathcal{M}$ be a matroid on ground set E with rank function r and let k be a positive integer. We say that a partition (X, Y) of E is a *vertical k-separation* if

$$\min\{r(X), r(Y)\} \geq k, \quad \text{and} \tag{2.1}$$

$$r(X) + r(Y) \leq r(E) + k - 1. \tag{2.2}$$

The *vertical connectivity* of $\mathcal{M}$, denoted by $\kappa(\mathcal{M})$, is defined to be the smallest integer j for which $\mathcal{M}$ has a j-separation. If $\mathcal{M}$ has no vertical separations at all, we let $\kappa(\mathcal{M}) = r(E)$. We say that $\mathcal{M}$ is *vertically h-connected* if $\kappa(\mathcal{M}) \geq h$ holds. The *Tutte connectivity* of $\mathcal{M}$, denoted by $\lambda(\mathcal{M})$, is defined analogously, except that in the definition of a k-separator (2.1) is replaced by $\min\{|X|, |Y|\} \geq k$. Hence vertical 2-connectivity is equivalent to Tutte 2-connectivity, while $\lambda(\mathcal{M}) \leq \kappa(\mathcal{M})$ holds in general. We refer the reader to [44] for more details on matroids and matroid connectivity.

This chapter was motivated by the following question, posed by Brigitte and Herman Servatius [2, Problem 17].

Let G be a graph and $\mathcal{R}_d(G)$ its d-dimensional generic rigidity matroid. Is there a (smallest) constant k_d such that G is uniquely determined by $\mathcal{R}_d(G)$ provided that $\mathcal{R}_d(G)$ is k_d-connected?

The original question was formulated in terms of Tutte connectivity, but, as we shall see, it is more convenient and more general to work with vertical connectivity.

$\mathcal{R}_1(G)$ is isomorphic to the cycle matroid of G, which implies, by a theorem of H. Whitney [71], that $k_1 = 3$. A *k-vertex separation* of G is a pair (G_1, G_2) of edge-disjoint subgraphs of G each with at least $k+1$ vertices such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = k$. The graph is said to be *k-vertex-connected* if it has at least $k+1$ vertices and has no j-vertex separation for all $0 \leq j \leq k-1$.

In Sections 2.2 and 2.3 we shall prove that k_2 exists and provide an explicit bound $k_2 \leq 11$. Our results lead to further questions about vertical connectivity of rigidity matroids. In Section 2.4 we determine $\kappa(\mathcal{R}_d(K_n))$ for all pairs d, n, while in Section 2.5 we show that if G is highly vertex-connected then $\kappa(\mathcal{R}_2(G))$ is also high (which does not hold for $\lambda(\mathcal{R}_2(G))$).

It is a major open problem to find a good characterization for independence in d-dimensional rigidity matroids, for $d \geq 3$. Thus the higher dimensional versions of our problem are probably substantially harder.

Henceforth we shall assume that $d = 2$ and omit the subscripts referring to the dimension (except in Section 2.4). Moreover, by a k-connected graph (matroid) we shall always mean a k-vertex-connected graph (vertically k-connected matroid, respectively).

2.2 Highly connected graphs

We first show that if G is highly connected then its rigidity matroid uniquely determines G. We shall rely on the following three results.

Lemma 2.2.1 *[18, Theorem 4.7.2], [21, Lemma 3.1] Suppose that $\mathcal{R}(G)$ is 2-connected. Then G is redundantly rigid.*

Theorem 2.2.2 *[35, Theorem 2] Every 6-connected graph is redundantly rigid.*

Theorem 2.2.3 *[21, Theorem 3.2] Suppose that G is 3-connected and redundantly rigid. Then $\mathcal{R}(G)$ is 2-connected.*

The proof method of our first result is motivated by a proof for (a special case of) Whitney's theorem, due to J. Edmonds (see [44]). Let $J \subseteq E$ be a set of elements in matroid $\mathcal{M}$. We say that J is a 2-*hyperplane* of $\mathcal{M}$ if $r(J) = r(E) - 2$ and J is closed, that is, for all $e \in E - J$ we have $r(J + e) = r(E) - 1$.

Theorem 2.2.4 *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If G is 7-connected then G is isomorphic to H.*

Proof. We say that a 2-hyperplane J of $\mathcal{R}(G)$ is 2-connected if the matroid restriction of $\mathcal{R}(G)$ to J is 2-connected. Since G is 7-connected, Theorems 2.2.2 and 2.2.3 imply that G is rigid and $E(G - v)$ (i.e. the edge set E minus the vertex bond of v) is a 2-connected 2-hyperplane of $\mathcal{R}(G)$ for all $v \in V(G)$.

Now consider an arbitrary 2-connected 2-hyperplane J of $\mathcal{R}(G)$. By Lemma 2.2.1 the subgraph $L = (V(J), J)$ of G on the set of end vertices of J is rigid. Thus $r(J) = 2|V(J)| - 3$ and, since 2-hyperplanes are closed sets, it follows that L is an induced subgraph of G. By using the fact that G is rigid, we obtain $|V(G)| = |V(J)| + 1$. Thus the complement of J corresponds to a vertex bond of G.

It follows that there is a bijection between $V(G)$ and the 2-connected 2-hyperplanes of $\mathcal{R}(G)$ and that $\mathcal{R}(G)$ uniquely determines the vertex-edge incidencies in G.

By the assumption of the theorem $\mathcal{R}(G)$ and $\mathcal{R}(H)$ are isomorphic. It follows from Theorems 2.2.2 and 2.2.3 that $\mathcal{R}(G)$ is 2-connected. Thus $\mathcal{R}(H)$ is also 2-connected and hence H is rigid by Lemma 2.2.1. This implies that $2|V(G)| - 3 =$

$r(G) = r(H) = 2|V(H)| - 3$ and hence $|V(G)| = |V(H)|$. Thus $\mathcal{R}(H)$ has $|V(H)|$ 2-connected 2-hyperplanes. So G and H are isomorphic, as claimed. $\square$

2.2.1 Examples

The bound on the connectivity of G in Theorem 2.2.4 could perhaps be improved to 6, but it cannot be replaced by 5. To see this consider the following example. Let G be a complete graph on six vertices. Split every vertex of G into 5 vertices of degree one, and identify these 5 vertices with the vertices of a complete graph K_5 or with five vertices of an arbitrary 5-connected graph H. It is easy to see that the resulting graph G' is 5-connected. See Figure 2.1 for two (non-isomorphic) examples, where K_5 is used five times, and for the remaining vertex H is chosen to be K_7 minus an edge.

By using the Henneberg inductive construction to verify independence, one can easily show that the graphs of Figure 2.1 are both rigid, in which all cross edges (i.e. edges corresponding to the edges of G) are bridges in their rigidity matroids. Thus the rigidity matroid of both graphs is isomorphic to the direct sum of five copies of $\mathcal{R}(K_5)$, one copy of $\mathcal{R}(K_7 - e)$, and fifteen copies of $\mathcal{R}(K_2)$. On the other hand, the two graphs are clearly non-isomorphic, since their degree sequences are different.

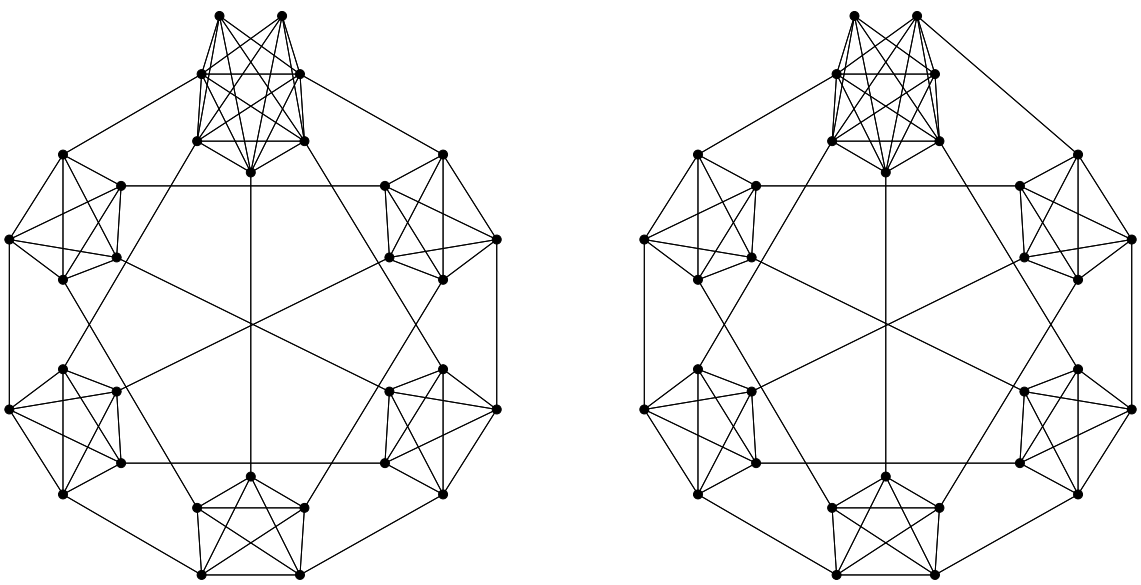


Figure 2.1: Two non-isomorphic rigid 5-connected graphs with isomorphic rigidity matroids.

2.3 Highly connected matroids

In this section we show that highly connected rigidity matroids have unique underlying graphs. We shall need Theorem 2.2.4 and the following two lemmas. Let $d(v)$ denote the degree of vertex v in G and let $\delta(G) = \min\{d(v) : v \in V(G)\}$ denote the minimum degree of G.

Lemma 2.3.1 *Let $G = (V, E)$ be a rigid graph on at least three vertices and suppose that $\mathcal{R}(G)$ is k-connected for some $k \geq 1$. Then $\delta(G) \geq k + 1$.*

Proof. Since G is rigid, G is 2-connected and $\delta(G) \geq 2$. Let X be the set of edges obtained from the star of some vertex v (i.e. from the set of edges incident with v) of degree $d(v)$ by deleting an arbitrary edge. Let $Y = E - X$. The 2-connectivity of G implies that (V, Y) is connected, and hence $r(Y) \geq |V| - 1 \geq d(v)$. Thus $\min\{r(X), r(Y)\} \geq d(v) - 1$ holds. Since X is a co-circuit of $\mathcal{R}(G)$, we have

$$r(X) + r(Y) \leq d(v) - 1 + r(E) - 1 = r(E) + d(v) - 2.$$

Hence (X, Y) is a $(d(v) - 1)$-separator of $\mathcal{R}(G)$, which implies $\delta(G) \geq k + 1$, as required. □

Lemma 2.3.2 *Let $G = (V, E)$ be a graph and suppose that $\mathcal{R}(G)$ is $(2k - 3)$-connected for some $k \geq 3$. Then G is k-connected.*

Proof. The hypothesis of the lemma implies that $\mathcal{R}(G)$ is 2-connected. Thus G is rigid by Lemma 2.2.1. Hence $r(E) = 2|V| - 3$ and, by Lemma 2.3.1, we have $\delta(G) \geq 2k - 2$ and $|V| \geq 2k - 1 \geq k + 1$.

For a contradiction suppose that G has a j-vertex separation (G_1, G_2) for some $j \leq k - 1$. Let $X = E(G_1)$ and $Y = E(G_2)$. Since $\delta(G) \geq 2k - 2$, we must have $\min\{r(X), r(Y)\} \geq 2k - 2$. By using (**??**) we can now deduce that $r(X) + r(Y) \leq 2|V(G_1)| - 3 + 2|V(G_2)| - 3 = 2(|V| + j) - 6 \leq 2|V| + 2k - 8 = r(E) + 2k - 5$. Hence (X, Y) is a $(2k - 4)$-separator of $\mathcal{R}(G)$, a contradiction. This proves the lemma. □

The main result of this section is now a direct corollary of Theorem 2.2.4 and Lemma 2.3.2.

Theorem 2.3.3 *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If $\mathcal{R}(G)$ is 11-connected then G is isomorphic to H.*

As we remarked earlier, Theorem 2.3.3 is valid for Tutte connectivity, too, since $\lambda(\mathcal{R}(G)) \leq \kappa(\mathcal{R}(G))$. Theorem 2.3.3 implies that $k_2 \leq 11$. It is not difficult to construct a pair of non-isomorphic rigidity circuits (i.e. graphs whose edges sets are circuits in their rigidity matroids) with the same number of edges. By such an example we have $k_2 \geq 3$.

2.4 The connectivity of $\mathcal{R}_d(K_n)$

One question which remains open after Theorem 2.3.3 is whether there exist graphs with arbitrarily highly connected rigidity matroids. In this section we give an affirmative answer by determining $\kappa(\mathcal{R}_d(K_n))$ for all pairs d, n. We shall use the following simple lemma. Part (i) is just one version of the Vertex Addition Lemma [68, Lemma 11.1.1], while part (ii) follows from the Edge Split Theorem [68, Theorem 11.1.7]. For a graph G let r_d denote the rank function of $\mathcal{R}_d(G)$.

Lemma 2.4.1 *Let G be a graph and let G' be obtained from G by adding a new vertex v and j new edges incident with v.*
(i) If $j \leq d$ then $r_d(G') = r_d(G) + j$,
(ii) If v is connected to all vertices of G and G is a non-rigid graph with $|V(G)| \geq d+1$ then $r_d(G') \geq r_d(G) + d + 1$.

The edge set of K_n is independent in $\mathcal{R}_d(K_n)$ if $n \leq d+1$. Thus it suffices to consider the case when $n \geq d+2$.

Theorem 2.4.2 *Let $K_n = (V, E)$ be a complete graph on n vertices with $n \geq d+2$. Then*

$$\kappa(\mathcal{R}_d(K_n)) = n - d.$$

Proof. For simplicity let r denote the rank function of $\mathcal{R}_d(K_n)$. Since K_n is rigid and $n \geq d+2$, we have $r(E) = dn - \binom{d+1}{2}$. First we show that $\mathcal{R}_d(K_n)$ has no $(n-d-1)$-separations. To this end consider a "red-blue 2-coloring", that is, a partition (R, B) of E, with $\min\{r(R), r(B)\} \geq n-d-1$ and suppose, for a contradiction, that

$$r(R)+r(B) \leq r(E)+n-d-2 = dn - \binom{d+1}{2} + n - d - 2 = (d+1)n - \binom{d+2}{2} - 1. \tag{2.3}$$

Let $G_R = (V, R)$ and $G_B = (V, B)$ denote the corresponding subgraphs of K_n. Clearly, these subgraphs are both non-rigid.

Let $W \subseteq V$ be a largest set of vertices with

$$r(R|W) + r(B|W) \geq (d+1)|W| - \binom{d+2}{2}, \tag{2.4}$$

where $R|W$ (resp. $B|W$) denotes the set of red (blue) edges induced by W. Since the sets R, B are non-empty, K_n has a bicolored complete graph on $d+2$ vertices, which is a circuit in $\mathcal{R}_d(K_n)$ and hence it satisfies (2.4). Thus W exists and we have $|W| \geq d+2$. On the other hand we cannot have $W = V$ by (2.3).

Consider a vertex $v \in V - W$. The edges incident with v must have the same color, since otherwise $r(R|(W+v)) + r(B|(W+v)) \geq r(R|W) + r(B|W) + d + 1 \geq (d+1)(|W|+1) - \binom{d+2}{2}$ would follow by Lemma 2.4.1(i), contradicting the maximality of W. Now suppose that v is incident with, say, red edges only. Then $G_R[W]$ is rigid, for otherwise $r(R|(W+v)) \geq r(R|W) + d + 1$ would hold by Lemma 2.4.1(ii), contradicting the maximality of W as above. Similarly, if v is incident with blue edges only then $G_B[W]$ must be rigid.

Suppose that each vertex in $V - W$ is incident with, say, red edges only. Then $G_R[W]$ is rigid, and hence $G_R[V]$ is also rigid by Lemma 2.4.1(i), a contradiction. Hence we may suppose that $V - W$ contains at least one vertex incident with red, and also at least one vertex incident with blue edges only. Then $G_R[W]$ and $G_B[W]$ are both rigid. Hence for any vertex $v \in V - W$ we have

$$r(R|(W+v)) + r(B|(W+v)) = 2d|W| - 2\binom{d+1}{2} + d \geq (d+1)(|W|+1) - \binom{d+2}{2},$$

which contradicts the maximality of W. It follows that $\mathcal{R}_d(K_n)$ has no $(n-d-1)$-separations, as claimed.

To complete the proof of the theorem we show that $\mathcal{R}_d(K_n)$ has no k-separations for $k \leq n-d-2$. Suppose that (R, B) is a partition of E with $\min\{r(R), r(B)\} = k \leq n-d-2$ and

$$r(R) + r(B) \leq r(E) + k - 1 = dn - \binom{d+1}{2} - 1 + k. \tag{2.5}$$

We may assume, by symmetry, that $r(R) = k$. Since $r(E) = dn - \binom{d+1}{2}$, there is a set $M \subset B$ with $|M| = n-d-1-k$ for which $r(R \cup M) = n-d-1$. Let $R' = R \cup M$ and $B' = B - M$. We also have $r(B) \geq r(B') \geq dn - \binom{d+1}{2} - (n-d-1) \geq n-d-1$, and hence $\min\{r(R'), r(B')\} = n-d-1$. By using (2.5) we can now deduce that

$$r(R') + r(B') \leq r(R) + (n-d-1-k) + r(B) \leq dn - \binom{d+1}{2} - 1 + k + (n-d-1-k).$$

Thus (R', B') is an $(n-d-1)$-separation in $\mathcal{R}_d(K_n)$, a contradiction. This implies $\kappa(\mathcal{R}_d(K_n)) \geq n-d$. To see that equality holds observe that by partitioning E into a set X of $n-d$ edges of a star of some vertex v, and its complement $Y = E - X$, we obtain an $(n-d)$-separation of $\mathcal{R}_d(K_n)$. $\square$

Since vertical connectivity is monotone with respect to edge deletions, Theorem 2.4.2 implies the existence of d-dimensional rigidity matroids with vertical connectivity l for all pairs of positive integers l, d.

2.5 Highly connected graphs revisited

One may also wonder whether a kind of converse of Lemma 2.3.2 holds. First observe that a highly connected graph G does not necessarily have a highly Tutte-connected rigidity matroid. The existence of a complete graph K_4 in G (whose edge set is a circuit in $\mathcal{R}(G)$) implies that $\lambda(\mathcal{R}(G)) \leq 6$, even if G is highly connected. On the other hand, as we shall show in this section, if G is highly connected then the vertical connectivity of its rigidity matroid must also be high. We need the following result of Lovász and Yemini which describes the rank funtion of the two-dimensional rigidity matroid of a graph G. A *cover* of G is a family of subgraphs $\{G_1, G_2, ..., G_p\}$ of G, where each G_i has at least two vertices and $E(G_1) \cup E(G_2) \cup ... \cup E(G_p) = E(G)$.

Theorem 2.5.1 *[35] Let $G = (V, E)$ be a graph. Then*

$$r(G) = \min \sum_{i=1}^{p} (2|V(G_i)| - 3),$$

where the minimum is taken over all covers $\{G_1, G_2, ..., G_p\}$ of G.

Let $G = (V, E)$ be a graph. We say that a partition (X, Y) of E is *essential* if (V, X) and (V, Y) are both non-rigid graphs.

Lemma 2.5.2 *Let $G = (V, E)$ be a k-connected graph, where $k \geq 6$, and let (X, Y) be an essential partition of E. Then*

$$r(X) + r(Y) \geq 2|V| + k - 6. \tag{2.6}$$

Proof. The proof is by induction on k. First we consider the case $k = 6$. For a contradiction suppose that there is a 6-connected graph $G = (V, E)$ with an essential partition (X, Y) of E for which

$$r(X) + r(Y) \leq 2|V| - 1. \tag{2.7}$$

We may assume that G has the least possible number n of vertices among such graphs and that G has the largest number of edges among such graphs on n vertices. We must have $n \geq 8$, because the only 6-connected graph on seven vertices is K_7, for which the lemma follows from Theorem 2.4.2[1].

[1] To see this observe that K_7 remains rigid after deleting any set $F \subset E(K_7)$ of edges with $r(F) \leq 3$. Hence an essential partition (X, Y) of $E(K_7)$ must satisfy $\min\{r(X), r(Y)\} \geq 4$. Therefore the violation of (2.6), that is, $r(X) + r(Y) \leq 13 = r(K_7) + 2$ would imply that (X, Y) is a 4-separation in K_7, contradicting Theorem 2.4.2.

By Theorem 2.5.1 there exists a cover $G_i = (V_i, E_i)$, $1 \le i \le p$, of (V, X) and a cover $G_j = (V_j, E_j)$, $p+1 \le j \le p+q$, of (V, Y), for which $\sum_{i=1}^{p}(2n_i - 3) = r(X)$ and $\sum_{i=p+1}^{p+q}(2n_i - 3) = r(Y)$, where $n_i = |V_i|$ for $1 \le i \le p+q$. Note that G_i is rigid for all $1 \le i \le p+q$. Hence

$$\sum_{i=1}^{p+q}(2n_i - 3) \le 2|V| - 1. \tag{2.8}$$

We may suppose, by the maximality of $|E|$, that $G[V_i]$ is complete, for $1 \le i \le p+q$.

Claim 2.5.3 *Every vertex of G occurs in at least two subgraphs G_i.*

Proof. Without loss of generality we may suppose that $v \in G_1$ but $v \notin G_i$ for all $2 \le i \le p+q$. Since G is 6-connected, $n_1 \ge 7$ must hold. Let $G' = G - v$, $G'_1 = G_1 - v$, $n'_1 = n_1 - 1$, and let $G'_i = G_i$, $n'_i = n_i$ for all $2 \le i \le p+q$. Let $S(v)$ denote the star of vertex v, i.e. the set of edges incident with v in G. Now $X' = X - S(v)$ and Y form a partition of the edge set of G'. We claim that this partition is essential. Let $V' = V - v$. If (V', X') is rigid then (V, X) is also rigid by Lemma 2.4.1(i), contradicting the fact that (X, Y) is an essential partition. If (V', Y) is rigid then, using the fact that (V, X) contains a rigid graph on at least seven vertices, we obtain $r(X) + r(Y) \ge r(G_1) + 2|V'| - 3 \ge 11 + 2|V| - 2 - 3 = 2|V| + 6$, contradicting (2.7). Now, since the G_i's cover X' and Y, respectively, we have

$$r(X') + r(Y) \le 2n'_1 - 3 + \sum_{i=2}^{p+q}(2n_i - 3) = \sum_{i=1}^{p+q}(2n_i - 3) - 2 \le 2|V| - 3 = 2|V'| - 1.$$

By the minimality of $|V|$ this implies that $G - v$ is not 6-connected. Thus there is a set of vertices $U \subseteq V - v$ with $|U| \le 5$ for which $G - v - U$ is disconnected. Since $G - U$ is connected, there must be two vertices a, b adjacent to v in G that are in different components of $G - v - U$. But, since $G[V_1]$ is complete, the neighbours of v are pairwise adjacent in G, which contradicts the fact that a and b are in different components of $G - v - U$. □

We can now proceed as in the proof of [35, Theorem 2]. Since G is 6-connected, we have $\sum_{v \in V_i}(n_i - 1) \ge 6$ for all $v \in V$. This inequality and Claim 2.5.3 imply that

$$\sum_{V_i \ni v}(2 - \frac{3}{n_i}) \ge 2$$

for all $v \in V$. Thus

$$\sum_{j=1}^{p+q}(2n_j - 3) = \sum_{j=1}^{p+q} n_j(2 - \frac{3}{n_j}) = \sum_{v \in V}\sum_{V_i \ni v}(2 - \frac{3}{n_i}) \ge 2|V|,$$

which contradicts (2.8). The statement of the lemma now follows for $k = 6$.

Next suppose $k \geq 7$ and that the statement of the lemma holds for every l-connected graph with $6 \leq l \leq k-1$. Suppose for a contradiction that there exists a k-connected graph $G = (V, E)$ with an essential partition (X, Y) of E violating (2.6). Let $v \in V$ be a vertex with $S(v) \cap X \neq \emptyset \neq S(v) \cap Y$. Since G is rigid, by Theorem 2.2.2, we have that X and Y are both non-empty, and hence v exists.

Consider the graph $G - v$ and the partition $(X - S(v), Y - S(v))$ of $E(G - v) = E - S(v)$. We claim that this partition is essential. By symmetry it suffices to show that $(V - v, X - S(v))$ is not rigid. Suppose that it is rigid. Then $|S(v) \cap X| = 1$ follows by Lemma 2.4.1(i), and we have

$$r(X) + r(Y) \geq r(X - S(v)) + r(S(v)) - 1 \geq 2|V - v| + k - 4 = 2|V| + k - 6,$$

which contradicts the assumption that (X, Y) is an essential partition of E violating (2.6).

By using the induction hypothesis, the choice of v, and that $G - v$ is $(k-1)$-connected we can now deduce that

$$r(X) + r(Y) - 3 \geq r(X - S(v)) + r(Y - S(v)) \geq 2|V'| + (k-1) - 6 = 2|V| + k - 9$$

which contradicts the assumption that (X, Y) is an essential partition of E violating (2.6). This completes the proof of the lemma. □

Theorem 2.5.4 *Let $G = (V, E)$ be a k-connected graph, where $k \geq 6$. Then $\kappa(\mathcal{R}(G)) \geq k - 2$.*

Proof. If X, Y is a vertical l-separation of the matroid $\mathcal{M}$ for some l then $\max\{r(X), r(Y)\} < r(\mathcal{M})$ must hold. Thus the theorem follows from Lemma 2.5.2. □

The lower bound of Theorem 2.5.4 is best possible for all $k \geq 6$. To see this take two disjoint k-connected graphs and connect them by k disjoint edges $e_1, e_2, ..., e_k$. Call the resulting graph $G = (V, E)$. It is easy to see that G is k-connected. Let $X = \{e_1, e_2, ..., e_{k-2}\}$ and $Y = E - X$. The partition (X, Y) is a $(k-2)$-separation of $\mathcal{R}(G)$, which shows that $\kappa(\mathcal{R}(G)) \leq k - 2$.

We also remark that Theorem 2.5.4 and the fact that the smallest bipartite graph whose two-dimensional rigidity matroid is a circuit is $K_{3,4}$ imply that the Tutte connectivity of $\mathcal{R}(K_{m,n})$ is at least 11, for all $m, n \geq 13$ (c.f. Theorem 2.3.3).

Chapter 3

Highly vertex-redundantly rigid graphs

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ if $|V| \geq k + 1$ and after deleting any set of at most $k - 1$ vertices the resulting graph is rigid in $\mathbb{R}^d$. A k-rigid graph G is called *minimally k-rigid* if the omission of an arbitrary edge results in a graph that is not k-rigid. B. Servatius showed that a 2-rigid graph in $\mathbb{R}^2$ has at least $2|V| - 1$ edges and this bound is sharp. We extend this lower bound for arbitrary values of k and d and show its sharpness for the cases where $k = 2$ and d is arbitrary and where $k = d = 3$. We also provide a sharp upper bound for the number of edges of minimally k-rigid graphs in $\mathbb{R}^d$ for all k.

3.1 k-rigid graphs

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$, or simply *$[k, d]$-rigid*, if $|V| \geq k + 1$ and, for any $U \subseteq V$ with $|U| \leq k - 1$, the graph $G - U$ is rigid in $\mathbb{R}^d$. In this context, we will call graphs that are rigid in $\mathbb{R}^d$ $[1, d]$-rigid. Every $[k, d]$-rigid graph is $[l, d]$-rigid by definition for $1 \leq l \leq k$. We remark that another equivalent definition of $[k, d]$-rigidity is also used in the literature. By this equivalent definition a graph is $[k, d]$-rigid if $|V| \geq k + 1$ and the deletion of any set of $k - 1$ vertices results in a graph that is rigid in $\mathbb{R}^d$. The following well-known lemma shows the equivalence of these two definitions.

Lemma 3.1.1 *A graph $G = (V, E)$, with $|V| \geq k + 1$, is $[k, d]$-rigid if and only if the deletion of any set of $k - 1$ vertices results in a graph that is rigid in $\mathbb{R}^d$.*

We will use both definitions.

G is called *minimally* $[k,d]$*-rigid* if it is $[k,d]$-rigid but $G-e$ fails to be $[k,d]$-rigid for every $e \in E$. G is said to be *strongly minimally* $[k,d]$*-rigid* if it is minimally $[k,d]$-rigid and there is no (minimally) $[k,d]$-rigid graph on the same vertex set with less edges. If G is minimally $[k,d]$-rigid but not strongly minimally $[k,d]$-rigid, then it is called *weakly minimally* $[k,d]$*-rigid.*

The investigation of $[k,d]$-rigid graphs was commenced in the plane by B. Servatius [52] and was continued recently in higher dimensions by Anderson, Montevallian, Summers and Yu [40, 41, 54, 55] motivated by multi-agent formations and sensor networks.

The following theorem gives a formula for the edge number of minimally rigid graphs.

Theorem 3.1.2 ([68]) *Let $G=(V,E)$ be minimally $[1,d]$-rigid. If $|V| \geq d+1$ then $|E| = d|V| - \binom{d+1}{2}$.*

We note that the proof of this theorem follows from the fact that the edge set of a minimally $[1,d]$-rigid graph corresponds to a base of the rigidity matroid of the graph. Hence it is not surprising that the edge sets of minimally $[1,d]$-rigid graphs on the same node set have the same cardinality. However, as we will see later, this is not true for $[k,d]$-rigid graphs when $k \geq 2$, there are minimally $[k,d]$-rigid graphs for all $k \geq 2$ and $d \geq 1$ with different edge numbers, that is, the set of weakly minimally $[k,d]$-rigid graphs is not empty for any d if $k \geq 2$.

To see a simple example, consider the case where $d=1$. G is rigid in $\mathbb{R}^1$ if and only if G is connected. Hence G is minimally $[k,1]$-rigid if and only if it is minimally k-connected. It is easy to construct minimally k-connected graphs with different edge-numbers, for example, the complete bipartite graph $K_{n-k,k}$ is minimally k-connected with $k(n-k)$ edges and there are k-regular k-connected graphs that must be minimal and have $kn/2$ edges.

It was shown by B. Servatius [52] that the smallest possible number of edges in a $[2,2]$-rigid graph is $2|V|-1$ and this bound is sharp. Later, lower and upper bounds were provided for the edge number of minimally $[k,d]$-rigid graphs for $d=2$ and 3 in [40, 41, 54, 55]). We summarize these results in the following theorem.

Theorem 3.1.3 *Let $G=(V,E)$ be a minimally $[k,d]$-rigid graph. Then*

(i) $|E| \geq 2|V|-1$ *if $k=d=2$ and $|V|$ is sufficiently large. This bound is sharp.*

(ii) $|E| \geq 2|V|+2$ *if $k=3$, $d=2$ and $|V|$ is sufficiently large. This bound is sharp.*

(iii) $|E| \geq \lceil \frac{k+1}{2}|V| \rceil$ *if* k *is arbitrary,* $d = 2$ *and* $|V|$ *is sufficiently large.*

(iv) $|E| = \Omega\left(\binom{k+d}{2}n\right)$ *if* k *is arbitrary and* $d = 2$ *or 3.*

The main result of this chapter is a sharp upper bound for the number of edges of minimally $[k, d]$-rigid graphs for every pair $[k, d]$. We provide a lower bound for the number of edges of minimally $[k, d]$-rigid graphs which is sharp for $k = 2$ for all d and for $k = 3,\ d \leq 3$. We also show that weakly minimally $[k, d]$-rigid graphs exist for every pair $[k, d]$ and we disprove a conjecture of Summers, Yu and Anderson [54, 55].

3.2 Preliminaries – Operations preserving rigidity

Constructive characterizations are useful tools in combinatorial rigidity. Even though we do not have a constructive characterization theorem for the class of rigid graphs for $d \geq 3$ it can be very useful to find operations that preserve rigidity. In this section we mention some of these operations.

The *d-dimensional Henneberg-0 extension*, or simply *0-extension*, on G adds a new vertex and connects it to d distinct vertices of G. The *d-dimensional 1-extension*, or simply *1-extension*, deletes an edge $uw \in E$, adds a new vertex v and connects it to u, v and $d-1$ other vertices of G. The d-dimensional 0-extension is also called *d-valent vertex addition* and the d-dimensional 1-extension is also called $d+1$-*valent edge split.*

Theorem 3.2.1 ([59]) *If* G *is rigid in* $\mathbb{R}^d$ *and* G' *is obtained from* G *by a* d-*dimensional 0-extension or 1-extension operation then* G' *is rigid in* $\mathbb{R}^d$.

As d-dimensional 0- and 1-extensions are used when we are in $\mathbb{R}^d$, we will simply call them 0- and 1-extensions if d is clear from context. There are some more operations that are known to preserve rigidity in higher dimensions. In this paper, we will use the following that we call a *(d-dimensional) simplex-based X-replacement.* Let $d \geq 2$ and let $a, b, w_1, \ldots, w_{d-2}$ be a complete subgraph of G and $cd \in E$ an edge which is node-disjoint from the simplex. The d-dimensional simplex-based X-replacement extension deletes ab, cd, adds a new vertex v and connects it to $a, b, c, d, w_1, \ldots, w_{d-2}$. When $d = 2$ or 3, we call a d-dimensional simplex-based X-replacement a *2-dimensional X-replacement* or a *triangle-based X-replacement*, respectively. It is folklore that these latter two operations preserve rigidity as the following lemma shows (in a more general way). For completeness, we give the proof of this lemma.

Lemma 3.2.2 *Let G be rigid in $\mathbb{R}^d$ and let G' be the result of a d-dimensional simplex-based X-replacement applied to G. Then G' is rigid in $\mathbb{R}^d$.*

Proof. In the proof, we will use special (non-generic) realizations of graphs. It is well-known that for a 0-extension we do not really need a generic realization, that is:

Claim 3.2.3 *Let (G, p) be independent in the d-dimensional rigidity matroid and let G' be the graph that arises from G by a d-dimensional 0-extension such that $V(G') = V(G) + v$ and let p' be a realization of G' in $\mathbb{R}^d$ such that $p(u) = p'(u)$ for every $u \in V$. Suppose that $p'(v)$ and its d neighbors have full affine span. Then (G', p') is independent in the d-dimensional rigidity matroid.*

We may assume that G is minimally rigid in $\mathbb{R}^d$ by deleting some redundant edges of G other than those we use for the extension. Let (G, p) be a generic realization of G. Let S be the hyperplane that contains the $(d-1)$ dimensional simplex spanned by $p(a), p(b), p(w_1), \ldots, p(w_{d-2})$ and let ℓ be the line of $p(c), p(d)$. Put $p(v) = S \cap \ell$ and let $G_0 = (V + v, E \cup \{va, vc\} \cup \{vw_i : 1 \leq i \leq d-2\})$. By Lemma 3.2.3 framework (G_0, p) is independent and hence minimally rigid.

Now we construct framework (G', p) from (G_0, p) by replacing edges ab and cd with vb and vd, respectively. We shall prove that (G', p) is rigid. First add vb, let $G_1 = G_0 + vb$. There is a unique M-circuit in (G_1, p) in the d-dimensional rigidity matroid which is the K_{d+1} induced by $v, a, b, w_1, \ldots, w_{d-2}$. (Note that points $p(v), p(a), p(b), p(w_1), \ldots, p(w_{d-2})$ lie on a hyperplane.) Thus with the notation $G_1 - ab = G_2$ framework (G_2, p) is independent.

Similarly, with the notation $G_3 = G_2 + vd$ the unique M-circuit in the the d-dimensional rigidity matroid of the framework (G_3, p) is the triangle spanned by v, c, d. Again, with removing cd we get an independent framework, equivalently (G', p) is rigid as we claimed. $\square$

3.3 The effect of coning on $[k, d]$-rigid graphs

We shall also use another type of operation that not only preserves rigidity of graphs but augments a $[1, d]$-rigid graph to a $[1, d+1]$-rigid one. The *cone graph* of G is the graph that arises from G by adding a new vertex v and edges vu for every $u \in V$. We will denote this graph by $G * v$. The operation that creates the cone graph of G is called *coning*.

Theorem 3.3.1 (Whiteley [65]) *A graph G is $[1,d]$-rigid if and only if the cone graph $G * v$ is $[1,d+1]$-rigid.*

Next, we prove some important consequences of Theorem 3.3.1 that will be useful throughout this paper. We refer the reader to [29] for the proof of the next lemma.

Lemma 3.3.2 *Let $e \in E$ be an M-bridge in $\mathcal{R}_d(G)$. Then e is a M-bridge in $\mathcal{R}_{d+1}(G * v)$.*

We remark that Theorem 3.3.1 cannot be generalized to k-rigid graphs. That is, if G is $[k,d]$-rigid for some $k \geq 2$, then $G * v$ is not necessarily $[k,d+1]$-rigid. For example, C_n is $[2,1]$-rigid, but $C_n * v$ (which is the wheel graph with $n+1$ vertices) is not $[2,2]$-rigid. However, the following results show that coning can be used to construct $[k,d]$-rigid graphs.

Lemma 3.3.3 *Let G be a $[k,d+1]$-rigid graph. Then G is $[k+1,d]$-rigid.*

Proof. Let G' be a $[1,d+1]$-rigid graph that we obtain from G by deleting $k-1$ arbitrary vertices. Suppose, for a contradiction, that there is a vertex $u \in V(G')$ such that $G' - u$ is not $[1,d]$-rigid. Then $(G' - u) * u$ is not $[1,d+1]$-rigid by Theorem 3.3.1 which contradicts the $[1,d+1]$-rigidity of $G' \subseteq (G' - u) * u$. □

Lemma 3.3.4 *Let $k \geq 2$ and $d \geq 1$ be integers and let $G = (V,E)$ be a $[k-1,d]$-rigid graph. Then $G * v$ is $[k,d]$-rigid.*

Proof. We need to show that, after deleting $k-1$ vertices, $G * v$ remains $[1,d]$-rigid. If v is omitted, then we are done by the $[k-1,d]$-rigidity of G. Otherwise, let $u_1, ..., u_{k-1}$ be the omitted vertices. $G - \{u_1, ..., u_{k-2}\}$ is $[1,d]$-rigid and v is connected to every neighbor of v_{k-1}. Hence $(G * v) - \{u_1, ..., u_{k-1}\}$ has a subgraph isomorphic to the $[1,d]$-rigid graph $G - \{u_1, ..., u_{k-2}\}$ showing that it is $[1,d]$-rigid. □

3.4 Lower bounds for the number of edges in $[k,d]$-rigid graphs

In this section, we present several lower bounds for the number of edges in $[k,d]$-rigid graphs for arbitrary positive integers k and d. Theorem 3.1.3 (i)-(iii) summarizes the lower bounds that were known earlier. First we extend (i) and (ii) to every dimension d.

Theorem 3.4.1 *If a graph $G = (V, E)$ is $[k, d]$-rigid with $|V| \geq d^2 + d + k$ then*

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k - 1 - \frac{d+1}{2} \right\rceil\right\}. \tag{3.1}$$

Note that the bound given in (3.1) coincides with the bounds given in Theorem 3.1.3 (i)-(ii) for $[k, d] = [2, 2],\ [3, 2]$ hence it is sharp for these values of k and d. In Sections 3.6 and 3.7, we show that this lower bound is sharp for $[k, d] = [2, d]$ where d is arbitrary, and for $[k, d] = [3, 3]$.

Proof. We prove this theorem by induction on k. For $k = 1$, the theorem immediately follows by Theorem 3.1.2.

Now, let $G = (V, E)$ be a $[k, d]$-rigid graph for $k \geq 2$ with $|V| \geq d^2 + d + k$ and assume that the theorem is true for $k - 1$. Let $v \in V$ be a node of maximum degree in G. As $G - v$ is $[k - 1, d]$-rigid with at least $d^2 + d + k - 1$ nodes,

$$|E(G - v)| \geq d(|V| - 1) - \binom{d+1}{2} + (k-2)d + \max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\}$$

by induction. Using this inequality, we have

$$\begin{aligned} |E| &\geq d(|V| - 1) - \binom{d+1}{2} + (k-2)d + \max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\} + \Delta(G) \\ &= d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\} + (\Delta(G) - 2d) \end{aligned}$$

Here, $\max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\} = 0 = \max\left\{0, \left\lceil k - 1 - \frac{d+1}{2} \right\rceil\right\}$ if $k - 1 \leq \frac{d+1}{2}$; and $\max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\} + 1 = \left\lceil k - 2 - \frac{d+1}{2} \right\rceil + 1 = \left\lceil k - 1 - \frac{d+1}{2} \right\rceil = \max\left\{0, \left\lceil k - 1 - \frac{d+1}{2} \right\rceil\right\}$ if $k - 1 > \frac{d+1}{2}$. Therefore, we need to prove that $\Delta(G) \geq 2d$ for all k and $\Delta(G) \geq 2d + 1$ also holds if $k - 1 > \frac{d+1}{2}$.

To prove that $\Delta(G) \geq 2d$ for all k, let us observe that if a graph $H = (V', E')$ is $[1, d]$-rigid with $|V'| \geq d^2 + d + 2$ then $\Delta(H) \geq 2d$. (To see this suppose that $\Delta(H) \leq 2d - 1$. Then $|E'| \leq |V'|d - \frac{|V'|}{2} < |V'|d - \binom{d+1}{2}$ which contradicts Theorem 3.1.2.) Since a $[k, d]$-rigid graph is also $[1, d]$-rigid and we have $|V| \geq d^2 + d + k$, we get that $\Delta(G) \geq 2d$. But then

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k - 2 - \frac{d+1}{2} \right\rceil\right\}$$

and hence $|E| > d|V|$ if $k - 1 > \frac{d+1}{2}$. Therefore, we get $\Delta(G) \geq 2d + 1$ if $k - 1 > \frac{d+1}{2}$ as we wanted. □

The following theorem gives a better lower bound if k is large compared to d. This result extends Theorem 3.1.3 (iii) for higher dimensions.

Theorem 3.4.2 *Let $k \geq d + 2$ and let $G = (V, E)$ be a $[k, d]$-rigid graph with $|V| \geq d + k$. Then $|E| \geq \left\lceil \frac{d+k-1}{2}|V| \right\rceil$.*

Proof. If we delete $k-1$ neighbors of a node v we get a $[1,d]$-rigid graph with at least $d+1$ nodes. Since the minimum degree of such a graph is at least d, we get $d_G(v) \geq k-1+d$. Thus the minimum degree in G is at least $k-1+d$ hence $|E| \geq \lceil \frac{d+k-1}{2}|V| \rceil$. $\square$

3.5 Upper bound for the number of edges in minimally $[k,d]$-rigid graphs

In this section, we give an upper bound for the number of edges of minimally $[k,d]$-rigid graphs. We refer to [29] for the proof of the following lemma.

Lemma 3.5.1 *Suppose that G is a minimally $[k,d]$-rigid graph. Then G is independent in $\mathcal{R}_{d+k-1}(G)$.*

By combining Lemma 3.5.1 and Theorem 3.1.2, we immediately get the following upper bound.

Theorem 3.5.2 *Let $G=(V,E)$ be a minimally $[k,d]$-rigid graph. Then*

$$|E| \leq (d+k-1)|V| - \binom{d+k}{2}.$$

The sharpness of this bound for $d \geq 2$ will be proved later in Lemma 3.7.4. As a graph is $[k,1]$-rigid if and only if it is k-connected Mader's sharp upper bound for the edge number of minimally k-connected graphs can be applied for the edge number of minimally $[k,1]$-rigid graphs, see [36]. This gives us the following.

Theorem 3.5.3 *Let $G=(V,E)$ be a minimally $[k,1]$-rigid graph with $|V| \geq 3k-1$. Then*

$$|E| \leq k|V| - k^2$$

and this bound is sharp.

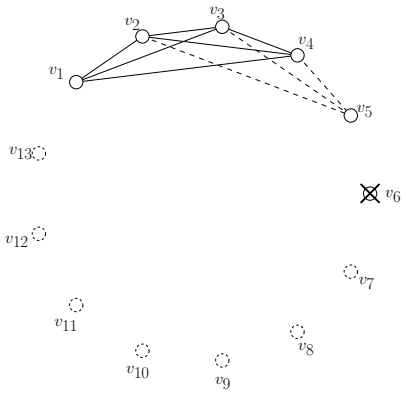
3.6 Minimally $[2,d]$-rigid graphs

In this section, we consider the case where $k=2$. First we give an example that shows the lower bound given in Theorem 3.4.1 is sharp for $k=2$ in any dimension and next we disprove Conjecture 3.6.3.

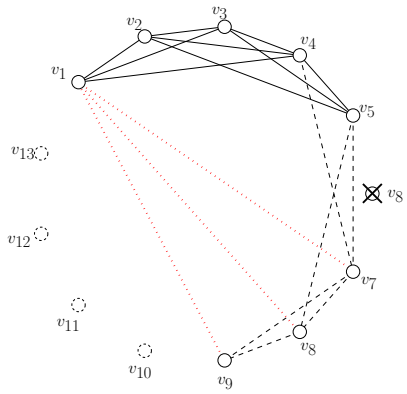
Consider graph C_n^d and its subgraph L_d induced by vertices $v_{n-d+1},\ldots,v_n$. (Note that L_d is isomorphic to K_d.) $H_{n,2}^d = C_n^d - E(L_d)$ denotes the graph we get from C_n^d after deleting the edge set of L_d. First we prove that $H_{n,2}^d$ is $[2,d]$-rigid.

Lemma 3.6.1 *$H^d_{n,2}$ is $[2,d]$-rigid if $n \geq 3d$.*

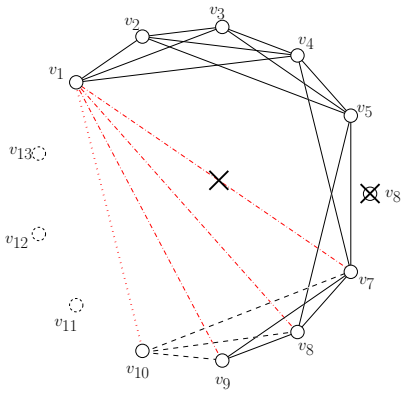
The proof of Lemma 3.6.1 is the generalization of the construction showed on Figure 3.1. See [29] for the details.



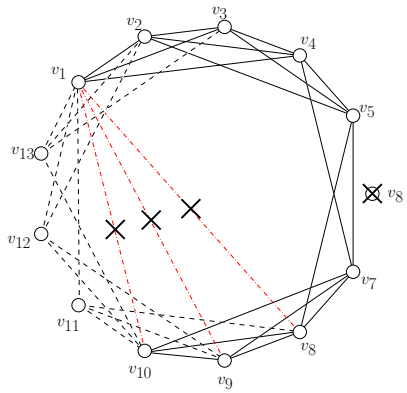
(a) Add v_5 with a 0-extension.



(b) Add v_7, v_8, v_9 with 0-extensions by adding the extra edges $v_j v_1$ for $7 \leq j \leq 9$.



(c) Add v_{10} with 1-extension on the extra edge $v_7 v_1$ by adding the extra edge $v_{10} v_1$



(d) Add v_{11}, v_{12}, v_{13} with 1-extensions on the edges $v_8 v_1, v_9 v_1, v_{10} v_1$, respectively.

Figure 3.1: Building up $C^3_{13} - E(L_3) - v_5$ using Henneberg operations.

If $G = (V, E)$ is $[2,d]$-rigid then $|E| \geq d|V| - \binom{d+1}{2} + d = d|V| - \binom{d}{2}$ if $|V| \geq d^2 + d + 2$ by Theorem 3.4.1. $|E(H^d_{n,2})| = dn - \binom{d}{2}$ since C^d_n has dn edges if $n \geq 2d+1$ and the deleted edges form a complete subgraph with d vertices. Hence by Lemma 3.6.1 we get the main result of this section:

Theorem 3.6.2 *If $G = (V, E)$ is a strongly minimally $[2,d]$-rigid graph with $|V| \geq d^2 + d + 2$ then $|E| = d|V| - \binom{d}{2}$.*

3.6.1 A counterexample for a conjecture of Summers et al.

B. Servatius proved a constructive characterization theorem for the class of strongly minimally $[2,2]$-rigid graphs that only uses 1-extensions in [52]. As far as we know, finding an inductive construction for the class of minimally $[2,2]$-rigid graphs is an open problem. It was observed in [55] that the 2-dimensional X-replacement preserves minimally $[2,2]$-rigidity in specific cases. Summers, Yu and Anderson conjectured that the 3-valent vertex addition and the 2-dimensional X-replacement operations are sufficient to build up every minimally $[2,2]$-rigid graph with at least nine vertices.

Conjecture 3.6.3 ([54, 55]) *Let $G = (V, E)$ be a minimally $[2,2]$-rigid graph with at least nine vertices. Then there exists either (a) a degree 4 vertex on which a reverse X-replacement operation can be performed to obtain a minimal $[2,2]$-rigid graph or (b) there exists a degree three vertex on which a reverse 3-valent vertex addition can be performed to obtain a minimally $[2,2]$-rigid graph.*

Now we disprove Conjecture 3.6.3 by constructing minimally [2,2]-rigid graphs that do not have a vertex at which the reverse degree 3 vertex addition or the reverse X-replacement can be performed. To give such an example, we will need the following simple observation.

We define an operation called K_4-*extension* that preserves $[2,2]$-rigidity. Let $G = (V, E)$ be a graph with $|V| \geq 4$, and let $v_1, v_2, v_3, v_4 \in V$ be four distinct vertices. The K_4-extension adds four new vertices u_1, u_2, u_3, u_4 to G, connects v_i to u_i for every $1 \leq i \leq 4$ and u_k to u_l for every pair $1 \leq k, l \leq 4$.

Claim 3.6.4 *If $G = (V, E)$ is $[2,2]$-rigid then $G' = (V', E')$ obtained by a K_4-extension is also $[2,2]$-rigid. Furthermore $G' - e$ is not $[2,2]$-rigid for any $e \in E' - E$.*

Proof. Clearly, $G' - v$ is rigid for any $v \in V'$. Consider the graph $G' - e$ for some $e \in E' - E$. Let $u_i \in V' - V$ be such that e is not incident to u_i. We claim that $G'' = G' - u_i - e$ is not rigid. G'' consist of G and a set of three vertices that is incident to five edges only. Hence there are only $2|V| - 3 + 5 = 2|V'| - 4$ independent edges in G'' thus G'' is not rigid as we claimed. $\square$

Now let $G_0 = (V_0, E_0)$ be a $[2,2]$-rigid graph with $V_0 \geq 4$. Apply some K_4-extensions to vertices of V_0, let the resulting graph be $G_1 = (V_1, E_1)$ (see Figure 3.2). Suppose that every vertex in V_0 is incident to at least five edges from $E_1 - E_0$. After the extensions, delete edges from E_1 (if necessary) to obtain a minimally $[2,2]$-rigid graph $G_2 = (V_1, E_2)$. By Claim 3.6.4, deleting any edge from $E_1 - E_0$ results

in a graph that is not $[2,2]$-rigid hence the minimum degree in G_2 is four and all the degree four vertices are in $V_1 - V_0$. Clearly we cannot perform the reverse degree 3 vertex addition in G_2. Every vertex in $V_1 - V_0$ is contained in a K_4 subgraph of G_2 and every reverse X-replacement on one of these vertices creates a parallel pair of edges. Thus no reverse X-replacement operation preserves minimal $[2,2]$-rigidity of G_2. This disproves Conjecture 3.6.3.

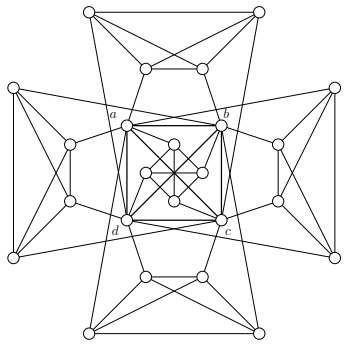


Figure 3.2: A counterexample H for Conjecture 3.6.3 that we get by performing five K_4-extensions on the subgraph induced by vertices a, b, c, d. K_4 is minimally $[2,2]$-rigid hence H is $[2,2]$-rigid by Claim 3.6.4. It can be easily seen that deleting any of the edges bc, cd, db from graph $H - a$ results in a non-rigid graph. By symmetry, the deletion of any edge of the starting graph results in a graph that is not $[2,2]$-rigid. This implies that G_c is minimally $[2,2]$-rigid.

We remark that for any positive integer t graph G_1 can be constructed such that every vertex in V_0 is incident to at least t edges from $E_1 - E_0$. Hence the minimum degree in G_2 is four and the vertices in V_0 have degree at least t. Since t can be arbitrarily large this example shows that it may not be easy to find a constructive characterization that only uses operations that add low-degree vertices.

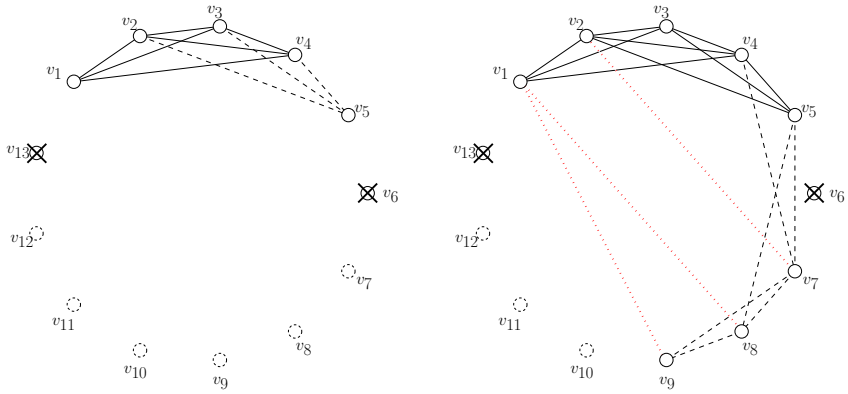
3.7 Strongly minimally $[3,3]$-rigid graphs

In this section, we give an example that shows that the lower bound given in Theorem 3.4.1 is sharp when $k = d = 3$.

Lemma 3.7.1 *C_n^3 is $[3,3]$-rigid if $n \geq 9$.*

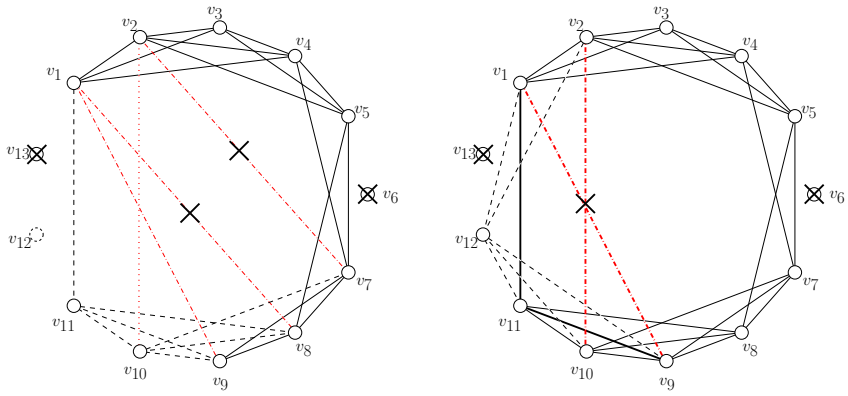
The proof of Lemma 3.6.1 is the generalization of the construction showed on Figure 3.3. See [29] for the details.

We have proved that C_n^3 is $[3,3]$-rigid. It is easy to see that C_n^3 has $3n$ edges if $n \geq 7$. These together with Theorem 3.4.1 gives the following:



(a) Add u_5 with a 0-extension.

(b) Add v_7, v_8, v_9 with 0-extensions by adding extra edges $v_{i+m}v_{\sigma(i)}$.



(c) Add v_{10} and v_{11} with 1-extensions on edges v_7v_2, v_8v_1, respectively. By performing the first of these extensions we create the extra edge $v_{10}v_2$.

(d) Add v_{12} with triangle-based X-replacement on edges v_2v_{10}, v_1v_9. (Note that $v_1v_9v_{11}$ is a triangle.)

Figure 3.3: Building up $C_{12}^3 - \{u, v\}$.

Theorem 3.7.2 *If $G = (V, E)$ is a strongly minimally $[3, 3]$-rigid graph with $|V| \geq 15$, then $|E| = 3|V|$.*

3.7.1 Examples for minimally $[k, d]$-rigid graphs

The question whether weakly minimally $[k, d]$-rigid graphs exist for every pair $[k, d]$ can still be solved without knowing the edge count of strongly minimally $[k, d]$-rigid graphs. There are examples for weakly minimally $[2, 2]$-rigid graphs in $[52, 54, 55]$ but the existence of weakly minimally $[k, d]$-rigid graphs for other values of k and d was open so far. In this section, we will give examples for minimally $[k, d]$-rigid graphs with the same number of vertices but with different number of

edges. Such a pair of graphs shows that the graph with the larger number of edges has to be weakly minimally $[k,d]$-rigid.

Let $H^d_{n,i}$ denote the cone graph of $H^d_{n,(i-1)}$ for $i \geq 3$. (For the definition of $H^d_{n,2}$ see Section 3.6.) By Lemma 3.3.4 and Lemma 3.6.1, we can get get a minimally $[k,d]$-rigid graph by deleting some edges of $H^d_{t,k}$ (to obtain minimality).

Corollary 3.7.3 *Let n, d and k be three positive integers such that $t \geq 3d$ and $k \geq 2$. Then there exists a minimally $[k,d]$-rigid graph $H^d_{t,k,\text{reduced}}$ with $n = t+k-2$ vertices and at most $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2)$ edges.*

We shall use Lemma 3.3.3 in the proof of the following lemma that also shows that the upper bound given in Theorem 3.5.2 is sharp for $d \geq 2$.

Lemma 3.7.4 *Let $t \geq 1$, $k \geq 1$ and $d \geq 2$ be three integers. There exists a minimally $[k,d]$-rigid graph with $n = t+k+d-1$ vertices and $(k+d-1)n - \binom{k+d}{2}$ edges.*

Proof. Define graph Y^c_t as follows for any integers c and t. Take the disjoint union of an independent set I_t of t nodes (on the vertex set $\{v_1,\ldots,v_t\}$) and a complete graph K_c (on the vertex set $\{w_1,\ldots,w_c\}$) and add edges v_iw_j for every pair $1 \leq i \leq t, 1 \leq j \leq c$ (see Figure 3.4).

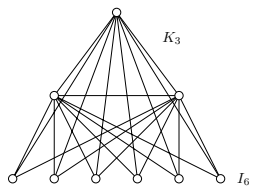


Figure 3.4: Y^3_6.

Y^1_t is minimally $[1,1]$-rigid as it is a tree. Hence by Theorem 3.3.1, we get that Y^c_t is minimally $[1,c]$-rigid as we get this graph after using the coning operation $c-1$ times on Y^1_t. Thus Y^{k+d-1}_t is $[1,k+d-1]$-rigid and hence it is $[k,d]$-rigid by Lemma 3.3.3.

Next we show that Y^{k+d-1}_t is *minimally* $[k,d]$-rigid. We have seen this for $k=1$. Now let $k,d \geq 2$. Let $uv \in E(Y^{k+d-1}_t)$ be an arbitrary edge. By symmetry, we can assume that $u,v \in \{v_1,v_2,w_1,w_2\}$. Observe that, after the omission of the $k-1$ nodes $v_{d+1},\ldots,v_{k+d+1}$ from Y^{k+d-1}_t, we get Y^d_t that is a minimally $[1,d]$-rigid graph as we observed before. Since $d \geq 2$, $uv \in E(Y^d_t)$ also holds. But $Y^d_t - uv$ is not $[1,d]$-rigid by the minimally $[1,d]$-rigidity of Y^d_t, hence $Y^{k+d-1}_t - uv$ is not $[k,d]$-rigid. Therefore, Y^{k+d-1}_t is minimally $[k,d]$-rigid.

Clearly, $|V(Y_t^{k+d-1})| = t+k+d-1 =: n$ and $|E(Y_t^{k+d-1})| = \binom{k+d-1}{2}+(k+d-1)t = (k+d-1)(t+k+d-1) - (k+d-1)^2 + \binom{k+d-1}{2} = (k+d-1)n - \binom{k+d}{2}$. $\square$

Corollary 3.7.5 *The upper bound given in Theorem 3.5.2 is sharp for all pair $[k,d]$ with $k,d \geq 2$.*

Some other examples for minimally $[k,d]$-rigid graphs can be found in a preliminary version of this paper (see [28]). Now, we are ready to prove the following theorem.

Theorem 3.7.6 *Let d and k be positive integers with $k \geq 2$. Then there are weakly minimally $[k,d]$-rigid graphs, that is, there are minimally $[k,d]$-rigid graphs that are not strongly minimally $[k,d]$-rigid.*

Proof. We only prove the theorem for $d \geq 2$ as we have seen in the Introduction that there are weakly minimally $[k,1]$-rigid graphs. By Corollary 3.7.3, there exists a minimally $[k,d]$-rigid graph on n nodes with at most $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2)$ edges if $n \geq 3d+k-2$. By Lemma 3.7.4, $Y_{n-k-d+1}^{k+d-1}$ is a minimally $[k,d]$-rigid graph on n nodes with at most $(k+d-1)n - \binom{k+d}{2}$ edges if $n \geq k+d$. Since $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2) < (k+d-1)n - \binom{k+d}{2}$ if n is sufficiently large, $Y_{n-k-d+1}^{k+d-1}$ is a weakly minimally $[k,d]$-rigid graph for all pair $[k,d]$ with $k,d \geq 2$ if n is sufficiently large. $\square$

3.8 Related problems

The results presented in this chapter are about the edge numbers of minimally $[k,d]$-rigid graphs. Similar questions were asked about minimally globally $[k,d]$-rigid graphs in [41, 54] where $G = (V,E)$ is globally $[k,d]$-rigid if $|V| \geq k+1$ and after deleting any set of at most $k-1$ vertices the resulting graph is globally rigid in $\mathbb{R}^d$. Other version of the problem is $[k,d]$-edge-rigidity (and global $[k,d]$-edge-rigidity) where instead of any set of at most $k-1$ vertices we delete any set of at most $k-1$ edges of the graph. Proving similar results on these variants of the problem is a possible direction of future research. Some of our methods (for example our lower bound for large k in Theorem 3.4.2) can be used easily for these graph classes. For example, as rigidity is a necessary condition for global rigidity, all our lower bounds are valid for globally $[k,d]$-rigid graphs. We note that a sharp upper bound for the edge number of minimally [2,2]-edge-rigid graphs was recently given by Jordán [24], as follows.

Theorem 3.8.1 (Jordán [24]) *Let $G = (V, E)$ be a minimally $[2, 2]$-edge-rigid simple graph with $|V| \geq 7$. Then*

$$|E| \leq 3|V| - 9.$$

The complete bipartite graph graph $K_{3,n-3}$ shows that this bound is sharp.

A different direction is to characterize inductively the class of graphs mentioned above for some values of $[k, d]$ which seems to be an interesting and difficult open question.

Chapter 4

Rigid two-dimensional frameworks with two coincident points

To verify the rigidity of (special families of) generic frameworks it is sometimes useful to consider non-generic realizations of graphs. For example, to prove a major conjecture of Tay and Whiteley [58], stating that a graph operation called *X-replacement* preserves rigidity in three-space, it could be useful to have a characterization of when a graph has an infinitesimally rigid realization in $\mathbb{R}^3$ in which the positions of four given vertices are coplanar, see [58, 61, 68].

Motivated by this connection, Jackson and Jordán [22] characterized when a graph has an infinitesimally rigid realization in $\mathbb{R}^2$ in which three given vertices are collinear. Recall that a set X of vertices in a minimally rigid graph G is *tight* if $i_G(X) = 2|X| - 3$. An *obstacle* for an ordered triple (x, y, z) of vertices is an ordered triple of tight sets (X, Y, Z) for which $X \cap Y = \{z\}$, $X \cap Z = \{y\}$, and $Y \cap Z = \{x\}$.

Theorem 4.0.2 *[22] Let $G = (V, E)$ be a minimally rigid graph and let $x, y, z \in V$ be distinct vertices. Then G has an infinitesimally rigid realization (G, p), in which $(p(x), p(y), p(z))$ are collinear if and only if G contains no obstacle for the triple (x, y, z).*

Watson [61] introduced the concept of flat realizations. He called a d-dimensional framework (G, p) *U-flat*, for some $U \subseteq V(G)$ with $2 \leq |U| \leq d + 1$, if the set $\{p(x) : x \in U\}$ is not affinely independent. He verified a number of results on U-flat realizations in $\mathbb{R}^3$ and formulated a conjecture for the existence of a two-dimensional U-flat realization. The special case when $|U| = 3$ is settled by Theorem 4.0.2 above. A slightly reformulated, but equivalent version of his conjecture for the case when $|U| = 2$ is as follows.

Conjecture 4.0.3 *[61, Conjecture 4.40] Let $G = (V, E)$ be a minimally rigid graph and $u, v \in V$ be two distinct vertices. Then there exists an infinitesimally rigid realization (G, p) of G in which $p(u) = p(v)$ if and only if*
(i) $uv \notin E$,
(ii) there is no $w \in V$ for which G contains an obstacle for $\{u, v, w\}$,
(iii) u and v have at most two common neighbours in G.

We have found a counterexample to Conjecture 4.0.3, see the graph of Figure 4.1.

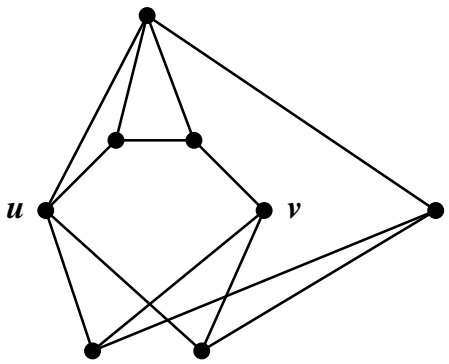


Figure 4.1: The graph G of this figure is minimally rigid and satisfies conditions (i)-(iii) of Conjecture 4.0.3 with respect to the designated vertex pair u, v. However, it does not have an infinitesimally rigid realization in which $p(u) = p(v)$. To see this observe that the existence of such a realization would imply that the graph obtained from G by contracting the vertex pair u, v is rigid (c.f. Theorem 4.2.6) but $G/\{u, v\}$ is not rigid.

Our main result in this chapter is a characterization for the existence of a two-dimensional U-flat realization for a given graph G and $U \subseteq V(G)$ with $|U| = 2$, which completes the solution of the two-dimensional flatness problem.

We need the following definitions. Let $G = (V, E)$ be a graph and let $u, v \in V$ be two distinct vertices of G. A realization (G, p) is called *uv-coincident* if $p(u) = p(v)$ holds. A uv-coincident realization is *uv-generic* if the set of coordinates of the points $\{p(z) : z \in V - v\}$ is algebraically independent over the rationals. Any two uv-coincident uv-generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the two-dimensional *uv-rigidity matroid* $\mathcal{R}_{uv}(G) = (E, r_{uv})$ of the graph G. We denote the rank of $\mathcal{R}_{uv}(G)$ by $r_{uv}(G)$. We say that the graph G is *uv-rigid* in $\mathbb{R}^2$ if $r_{uv}(G) = 2|V| - 3$ holds. A set $F \subseteq E$ is said to be *uv-independent* if F is independent in $\mathcal{R}_{uv}(G)$. The graph G is said to be *minimally uv-rigid* if G is uv-rigid and E is uv-independent.

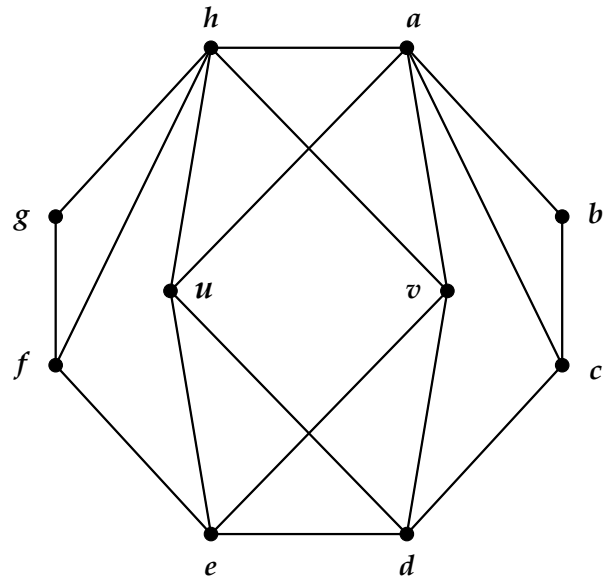


Figure 4.2: A rigid but not uv-rigid graph $G = (V, E)$ with $|V| = 10$. Consider the cover $\mathcal{K} = \{\{\{u, v, a, h\}, \{u, v, e, d\}\}, \{a, b, c\}, \{c, d\}, \{e, f\}, \{f, g, h\}\}$ of E. Its value equals 16, which is less than $2|V| - 3 = 17$ and hence G is not uv-rigid by Theorem 6.6.5 and Lemma 4.1.7.

4.1 The count matroid

Let $G = (V, E)$ be a graph and $u, v \in V$ be two distinct vertices of G. Let $\mathcal{H} = \{H_1, ..., H_k\}$ be a family with $H_i \subseteq V$, $1 \leq i \leq k$. We say that $\mathcal{H}$ is *uv-compatible* if $u, v \in H_i$ and $|H_i| \geq 3$ hold for all $1 \leq i \leq k$. We define the *value* of subsets of V of size at least two and of uv-compatible families as follows. For $H \subseteq V$ with $|H| \geq 2$ and $H \neq \{u, v\}$ we let

$$\mathrm{val}(H) = 2|H| - 3,$$

and put $\mathrm{val}(\{u, v\}) = 0$. For a uv-compatible family $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ we let

$$\mathrm{val}(\mathcal{H}) = \sum_{i=1}^{k} (2|H_i| - 3) - 2(k - 1).$$

Note that if $\mathcal{H} = \{H\}$ is a uv-compatible family containing only one set then the two definitions are compatible, i.e. $\mathrm{val}(\mathcal{H}) = \mathrm{val}(H)$ holds.

The value of a system $\mathcal{K} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_l\}$ of set families (which may consist of uv-compatible families as well as subsets of V) is defined by $\mathrm{val}(\mathcal{K}) = \sum_{i=1}^{l} \mathrm{val}(\mathcal{H}_i)$. Figure **??** shows an example for a cover containing a uv-compatible family.

The next lemmas will enable us to consider uv-compatible families of special types in the proof of Lemma 4.1.5 which is the main proof of this section.

Lemma 4.1.1 *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a uv-compatible family. If $|H_i \cap H_j| \geq 3$ for some pair $1 \leq i < j \leq k$, then there is a uv-compatible family $\mathcal{H}'$ with $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\mathcal{H}')$ for which $\mathrm{val}(\mathcal{H}') \leq \mathrm{val}(\mathcal{H}) - 1$.*

Proof. We may assume that $i = k-1$ and $j = k$. Let $\mathcal{H}' = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k)\}$. Then

$$\mathrm{val}(\mathcal{H}) = \sum_{l=1}^{k}(2|H_l| - 3) - 2(k-1) =$$

$$= \sum_{l=1}^{k-2}(2|H_l| - 3) - 2((k-1)-1) + (2|H_{k-1}| - 3) + (2|H_k| - 3) - 2 =$$

$$= \sum_{l=1}^{k-2}(2|H_l|-3)+(2|H_{k-1}\cup H_k|-3)-2((k-1)-1)+(2|H_{k-1}\cap H_k|-3)-2 \geq \mathrm{val}(\mathcal{H}')+1.$$

Clearly, we have $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\mathcal{H}')$. □

Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. We say that G is *uv-sparse* if for all $H \subseteq V$ with $|H| \geq 2$ we have $i_G(H) \leq \mathrm{val}(H)$ and for all uv-compatible families $\mathcal{H}$ we have $i_G(\mathcal{H}) \leq \mathrm{val}(\mathcal{H})$. Note that if G is uv-sparse then $uv \notin E$ must hold. A set $H \subseteq V$ of vertices with $|H| \geq 2$ (resp. a uv-compatible family $\mathcal{H} = \{H_1, \ldots, H_k\}$) is called *tight* if $i_G(H) = \mathrm{val}(H)$ (resp. $i_G(\mathcal{H}) = \mathrm{val}(\mathcal{H})$) holds.

Lemma 4.1.2 *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a uv-compatible family with $|H_i \cap H_j| = 2$ for all $1 \leq i < j \leq k$, and let $Y \subseteq V$ be a set of vertices with $|Y \cap \{u, v\}| \leq 1$ and $|Y \cap H_i| \geq 2$ for some $1 \leq i \leq k$. Then there is a uv-compatible family $\mathcal{H}'$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ for which $\mathrm{val}(\mathcal{H}') \leq \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$ holds. Furthermore, if G is uv-sparse and $\mathcal{H}$ and Y are both tight then $\mathcal{H}'$ is also tight.*

Proof. By renumbering the sets of $\mathcal{H}$, if necessary, we may assume that $|Y \cap H_i| \geq 2$ if $i \geq j$, for some $j \leq k$, and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq j-1$. Let $X = Y \cup \cup_{i=j}^{k} H_i$ and $\mathcal{H}' = \{H_1, \ldots, H_{j-1}, X\}$. With this notation

$$|X| = \sum_{i=j}^{k}|H_i| + |Y| - 2(k-j) - \sum_{i=j}^{k}|H_i \cap Y| + |Y \cap \{u, v\}|(k-j).$$

Then we have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ and

$$\mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) = \sum_{i=1}^{k}(2|H_i| - 3) - 2(k-1) + (2|Y| - 3) =$$

$$= \sum_{i=1}^{j-1}(2|H_i| - 3) - 2(j-1) + \sum_{i=j}^{k}(2|H_i| - 3) - 2(k-j) + (2|Y| - 3) =$$

$$= \sum_{i=1}^{j-1}(2|H_i|-3) + (2|X|-3) - 2(j-1) + 4(k-j) - 3(k-j+1) +$$

$$+2\sum_{i=j}^{k}|Y \cap H_i| - 2(k-j) - 2|Y \cap \{u,v\}|(k-j) \geq$$

$$\geq \mathrm{val}(\mathcal{H}') + \sum_{i=j}^{k}\mathrm{val}(Y \cap H_i).$$

Now suppose that $\mathcal{H}$ and Y are tight. Then we have

$$i(\mathcal{H}') + \sum_{i=j}^{k} i(Y \cap H_i) \geq i(\mathcal{H}) + i(Y) = \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) \geq$$

$$\geq \mathrm{val}(\mathcal{H}') + \sum_{i=j}^{k}\mathrm{val}(Y \cap H_i) \geq i(\mathcal{H}') + \sum_{i=j}^{k} i(Y \cap H_i),$$

where the first inequality follows from the fact that edges spanned by $\mathcal{H}$ or Y are spanned by $\mathcal{H}'$ and if some edge is spanned by both $\mathcal{H}$ and Y then it is spanned by $Y \cap H_i$ for some i. The first equality holds because $\mathcal{H}$ and Y are tight, and the second inequality holds by our calculations above. The last inequality holds because G is uv-sparse. Hence equality must hold everywhere, which implies that $\mathcal{H}'$ is also tight. □

Lemma 4.1.3 *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a uv-compatible family with $|H_i \cap H_j| = 2$ for all $1 \leq i < j \leq k$, and let $Y \subseteq V$ be a set of vertices with $Y \cap \{u,v\} = \emptyset$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$, for which $|Y \cap H_i| = |Y \cap H_j| = 1$ for some pair $1 \leq i < j \leq k$. Then there is a uv-compatible family $\mathcal{H}'$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ for which $\mathrm{val}(\mathcal{H}') = \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$. Furthermore, if G is uv-sparse and $\mathcal{H}$ and Y are both tight then $\mathcal{H}'$ is also tight.*

Proof. We may assume that $i = k-1$ and $j = k$. Let $\mathcal{H}' = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k \cup Y)\}$. Then

$$\mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) = \sum_{i=1}^{k}(2|H_i|-3) - 2(k-1) + (2|Y|-3) =$$

$$= \sum_{i=1}^{k-2}(2|H_i|-3) - 2((k-1)-1) - 2 + (2|H_{k-1}|-3) + (2|H_k|-3) + (2|Y|-3) =$$

$$= \sum_{i=1}^{k-2}(2|H_i|-3) - 2((k-1)-1) + (2(|H_{k-1}| + |H_k| + |Y|) - 3) - 8 =$$

$$= \sum_{i=1}^{k-2}(2|H_i| - 3) + (2|H_{k-1} \cup H_k \cup Y| - 3) - 2((k-1)-1) = \mathrm{val}(\mathcal{H}').$$

Clearly, we have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$.

Now suppose that G is uv-sparse and $\mathcal{H}$ and Y are tight. Then we have

$$i(\mathcal{H}) + i(Y) = \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) = \mathrm{val}(\mathcal{H}') \geq i(\mathcal{H}') \geq i(\mathcal{H}) + i(Y)$$

where the last inequality follows since $|Y \cap H_{k-1}| = |Y \cap H_k| = 1$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold everywhere, which implies that $\mathcal{H}'$ is also tight. □

Lemma 4.1.4 *Let $G = (V, E)$ be uv-sparse and let $X, Y \subseteq V$ be tight sets in G with $|X \cap Y| \geq 2$ and $X \neq \{u, v\} \neq Y$. Then $X \cap Y \neq \{u, v\}$ and $X \cup Y$ and $X \cap Y$ are also tight.*

Proof. If $X \cap Y \neq \{u, v\}$ then the lemma follows as in [22, Lemma 2.3]. Otherwise we obtain $i(\{u, v\}) = 1$, which contradicts the fact that G is uv-sparse. □

Lemma 4.1.5 *Let $G = (V, E)$ be uv-sparse and suppose that there is a tight uv-compatible family in G. Then there is a unique tight uv-compatible family $\mathcal{H}_{\max}$ in G for which $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\mathcal{H}_{\max})$ for all tight uv-compatible families $\mathcal{H}$ of G.*

Proof. It follows from Lemma 4.1.1 that if $\mathcal{H} = \{X_1, X_2, \ldots, X_k\}$ is a tight uv-compatible family in G then $X_i \cap X_j = \{u, v\}$ holds for all $1 \leq i < j \leq k$. Now consider a pair $\mathcal{H}_1 = \{X_1, X_2, \ldots, X_k\}$ and $\mathcal{H}_2 = \{Y_1, Y_2, \ldots, Y_l\}$ of tight uv-compatible families. Let $\mathcal{F} = (V, \mathcal{E})$ be a hypergraph where $\mathcal{E} = \{X_i - \{u, v\} : 1 \leq i \leq k\} \cup \{Y_j - \{u, v\} : 1 \leq j \leq l\}$ and let $C_1 = (V_1, \mathcal{E}_1), \ldots, C_t = (V_t, \mathcal{E}_t)$ be the connected components of $\mathcal{F}$. We define the following families:

$$\mathcal{H}_\cup = \{H_s : H_s = (\cup_{(X_i - \{u,v\}) \in \mathcal{E}_s} X_i) \cup (\cup_{(Y_j - \{u,v\}) \in \mathcal{E}_s} Y_j) \text{ for } 1 \leq s \leq t\}$$

$$\mathcal{H}_\cap = \{Z \subseteq V : |Z| \geq 3, \exists 1 \leq i \leq k, 1 \leq j \leq l \text{ such that } X_i \cap Y_j = Z\}$$

It is easy to see that $\mathcal{H}_\cup$ and $\mathcal{H}_\cap$ are both uv-compatible. For convenience we rename the families as $\mathcal{H}_\cup = \{A_1, \ldots, A_p\}$ and $\mathcal{H}_\cap = \{B_1, \ldots, B_q\}$. By using that $X_i \cap X_j = Y_{i'} \cap Y_{j'} = \{u, v\}$ we obtain $p + q \geq k + l$. We also have $i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap)$, since the family $\mathcal{H}_\cup$ spans all the edges spanned by $\mathcal{H}_1$ or $\mathcal{H}_2$ and $\mathcal{H}_\cap$ spans all the edges spanned by both $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus

$$\sum_{i=1}^{k}(2|X_i| - 3) - 2(k-1) + \sum_{j=1}^{l}(2|Y_j| - 3) - 2(l-1) = \mathrm{val}(\mathcal{H}_1) + \mathrm{val}(\mathcal{H}_2) =$$

$$= i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap) \leq \mathrm{val}(\mathcal{H}_\cup) + \mathrm{val}(\mathcal{H}_\cap) =$$

$$= \sum_{s=1}^{p}(2|A_s| - 3) - 2(p-1) + \sum_{t=1}^{q}(2|B_t| - 3) - 2(q-1) =$$

$$= \sum_{s=1}^{p} 2(|A_s| - 2) - (p-2) + \sum_{t=1}^{q} 2(|B_t| - 2) - (q-2) \leq$$

$$\leq \sum_{i=1}^{k} 2(|X_i| - 2) - (k-2) + \sum_{j=1}^{l} 2(|Y_j| - 2) - (l-2) =$$

$$= \sum_{i=1}^{k}(2|X_i| - 3) - 2(k-1) + \sum_{j=1}^{l}(2|Y_j| - 3) - 2(l-1),$$

where the last inequality follows from $\sum_{k=1}^{p}(|A_k|-2)+\sum_{l=1}^{q}(|B_l|-2) = \sum_{i=1}^{k}(|X_i|-2)+\sum_{j=1}^{l}(|Y_j|-2)$ and $p+q \geq k+l$. Hence we can deduce that $\mathcal{H}_\cup$ and $\mathcal{H}_\cap$ are both tight. Clearly, we have $\mathrm{cov}(\mathcal{H}_1) \cup \mathrm{cov}(\mathcal{H}_2) \subseteq \mathrm{cov}(\mathcal{H}_\cup)$. Thus the lemma follows by choosing the tight uv-compatible family $\mathcal{H}_{\max}$ of G for which $\mathrm{cov}(\mathcal{H}_{\max})$ is maximal. Note that the set of pairs of vertices covered by a tight uv-compatible family $\mathcal{H}$ uniquely determines $\mathcal{H}$ and hence $\mathcal{H}_{\max}$ is indeed unique. □

4.1.1 The matroid and its rank function

Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. In this subsection we prove that the family

$$\mathcal{I}_G = \{F : F \subseteq E, H = (V, F) \text{ is } uv\text{-sparse}\} \tag{4.1}$$

is a family of independent sets of a matroid on ground-set E. We shall also characterize the rank function of this matroid. We need the following definition.

Let $\mathcal{H} = \{X_1, \ldots, X_t\}$ be a uv-compatible family and let $H_1, \ldots, H_k$ be subsets of V of size at least two. We say that the system $\mathcal{K} = \{H_1, \ldots, H_k\}$ is *thin* if
(i) $|H_i \cap H_j| \leq 1$ for all pairs $1 \leq i, j \leq k$.
The system $\mathcal{L} = \{\mathcal{H}, H_1, \ldots, H_k\}$ is *thin* if (i) holds and
(ii) $X_i \cap X_j = \{u, v\}$ for all pairs $1 \leq i, j \leq t$, and
(iii) $|H_i \cap \cup_{j=1}^{t} X_j| \leq 1$ for all $1 \leq i \leq k$.

Theorem 4.1.6 *Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. Then $\mathcal{M}_{uv}(G) = (E, \mathcal{I}_G)$ is a matroid on ground-set E, where $\mathcal{I}_G$ is defined by (4.1). The rank of a set $E' \subseteq E$ in $\mathcal{M}_{uv}(G)$ is equal to*

$$\min\{\mathrm{val}(\mathcal{K}) : \mathcal{K} \ \textit{ is a thin cover of } E'\}.$$

Proof. Let $\mathcal{I} = \mathcal{I}_G$, let $E' \subseteq E$ and let $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \mathrm{val}(\mathcal{K})$ for all covers $\mathcal{K}$ of E'. We shall prove that there is a (thin) cover $\mathcal{K}$ of E' with $|F| = \mathrm{val}(\mathcal{K})$, from which the theorem will follow.

Let $J = (V, F)$ denote the subgraph induced by the edge set F. First suppose that there is no tight uv-compatible family in J and consider the following cover of F:

$$\mathcal{K}_1 = \{H_1, H_2, \ldots, H_k\},$$

where $H_1, H_2, \ldots, H_k$ are the maximal tight sets in J. Every edge $f \in F$ induces a tight set in J, hence $\mathcal{K}_1$ is indeed a cover of F. It is thin by Lemma 4.1.4. Thus

$$|F| = \sum_{j=1}^{k} |E_J(H_j)| = \sum_{j=1}^{k} (2|H_j| - 3) = \mathrm{val}(\mathcal{K}_1)$$

follows. We claim that $\mathcal{K}_1$ is a cover of E'. To see this consider an edge $ab = e \in E' - F$. Since F is maximal subset of E' in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. By our assumption there is no tight uv-compatible family in J, and hence there must be a tight set X in J with $a, b \in X$. Hence $X \subseteq H_i$ for some $1 \leq i \leq k$ which implies that $\mathcal{K}_1$ covers e, too.

Next suppose that there is a tight uv-compatible family in J and consider the following cover of F:

$$\mathcal{K}_2 = \{\mathcal{H}_{\max}, H_1, H_2, \ldots, H_k\},$$

where $\mathcal{H}_{\max} = \{X_1, X_2, \ldots, X_l\}$ is the uv-compatible family of G for which $\mathrm{cov}(\mathcal{H}_{\max})$ is maximal (c.f. Lemma 4.1.5) and $H_1, H_2, \ldots, H_k$ are maximal tight sets of $J' = (V, F - E(\mathcal{H}_{\max}))$. It is easy to see that $\mathcal{K}_2$ is indeed a cover of F. By Lemmas 4.1.1, 4.1.2, 4.1.3 and 4.1.4 the cover $\mathcal{K}_2$ is thin, and hence

$$|F| = \sum_{i=1}^{l} |E_J(X_i)| + \sum_{j=1}^{k} |E_J(H_j)| = \sum_{i=1}^{l} (2|X_i| - 3) - 2(l-1) + \sum_{j=1}^{k} (2|H_i| - 3) = \mathrm{val}(\mathcal{K}_2).$$

We claim that $\mathcal{K}_2$ is a cover of E'. As above, let $ab = e \in E' - F$ be an edge. By the maximality of F we have $F + e \notin \mathcal{I}$. Thus either there is a tight set $X \subseteq V$ in J with $a, b \in X$ or there is a tight uv-compatible family $\mathcal{H}' = \{Y_1, \ldots, Y_t\}$ in J with $a, b \in Y_i$ for some $1 \leq i \leq t$.

In the latter case Lemma 4.1.5 implies that $\mathrm{cov}(\mathcal{H}') \subseteq \mathrm{cov}(\mathcal{H}_{\max})$ and hence e is covered by $\mathcal{K}_2$. In the former case, when $a, b \in X$ for some tight set X in J we have two possibilities. First suppose that $|X \cap \cup_{i=1}^{l} X_i| \geq 2$. Then we can deduce that $X \subseteq X_i$ for some $1 \leq i \leq l$ by using Lemma 4.1.2 or 4.1.3 and the maximality of $\mathcal{H}_{\max}$, which implies that $\mathcal{K}_2$ covers e. Next suppose that $|X \cap \cup_{i=1}^{l} X_i| \leq 1$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq H_i$ for some $1 \leq i \leq k$, since every edge of J' induces

a tight set and every tight set is contained in a maximal tight set. Hence e is covered by $\mathcal{K}_2$, as claimed. $\square$

4.1.2 Independence

Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Let G_{uv} denote the graph obtained from G by contracting the vertex pair u, v into a new vertex z_{uv} (and deleting the resulting loops and parallel copies of edges). Given a realization (G_{uv}, p_{uv}) of G_{uv}, we obtain a uv-coincident realization (G, p) of G by putting $p(u) = p(v) = p_{uv}(z_{uv})$ and $p(x) = p_{uv}(x)$ for all $x \in V - \{u, v\}$. Furthermore, each vector in the kernel of $R(G_{uv}, p_{uv})$ determines a vector in the kernel of $R(G, p)$ in a natural way. It follows that

$$\dim \mathrm{Ker} R(G, p) \geq \dim \mathrm{Ker} R(G_{uv}, p_{uv}). \tag{4.2}$$

We can use this fact to prove that uv-independence implies independence in $\mathcal{M}_{uv}(G)$. The reverse implication will be verified in the next section.

Lemma 4.1.7 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. If G is uv-independent then E is independent in $\mathcal{M}_{uv}(G)$.*

Proof. Let (G, p) be an independent uv-coincident realization of G. Independence implies that $i(H) \leq \mathrm{val}(H)$ holds for all $H \subseteq V$ with $|H| \geq 2$. Since $p(u) = p(v)$,$uv \notin E$ follows.

Let $\mathcal{H} = \{X_1, \ldots, X_k\}$ be a uv-compatible family and consider the subgraph $F = (\cup_{i=1}^k X_i, \cup_{i=1}^k E(X_i))$. By contracting the vertex pair u, v in F we obtain the graph F_{uv}, in which $\mathcal{H}_{uv} = \{X_1/\{u, v\}, \ldots, X_k/\{u, v\}\}$ is a cover where $X_i/\{u, v\}$ denotes the set that we get from X_i by identifying u and v. Thus we get $r(F_{uv}) \leq \sum_{i=1}^k (2(|X_i|-1)-3)$ by using (the easy direction of) the Lovász-Yemini rank formula. This bound and (4.2) imply that $\dim \mathrm{Ker} R(F, p) \geq \dim \mathrm{Ker} R(F_{uv}, p_{uv}) \geq 2(|\cup_{i=1}^k X_i| - 1) - \sum_{i=1}^k (2|X_i| - 5)$. Since (G, p) is uv-independent, we have

$$i_F(\mathcal{H}) = |F| \leq 2\left|\bigcup_{i=1}^k X_i\right| - \left(2\left(|\bigcup_{i=1}^k X_i| - 1\right) - \sum_{i=1}^k (2|X_i| - 5)\right) =$$

$$\sum_{i=1}^k (2|X_i| - 3) - 2(k-1) = \mathrm{val}(\mathcal{H}).$$

Thus E is independent in $\mathcal{M}_{uv}(G)$, as claimed. $\square$

4.2 Inductive constructions

We shall need the following specialized versions of the two-dimensional Henneberg-extensions. Let $u, v \in V$ be two distinct vertices. The *0-uv-extension* operation is a 0-extension on a pair a, b with $\{a, b\} \neq \{u, v\}$. The *1-uv-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u, v\}$ is not a subset of $\{a, b, c\}$. The inverse operations are called *0-uv-reduction* and *1-uv-reduction*, respectively.

The Henneberg operations preserve independence in the two-dimensional rigidity matroid, see e.g. [68, Lemma 2.1.3, Theorem 2.2.2]. The same arguments can be used to verify the next lemma.

Lemma 4.2.1 *Let $G = (V, E)$ be an uv-independent graph and suppose that G' is obtained from G by a 0-uv-extension or a 1-uv-extension. Then G' is uv-independent.*

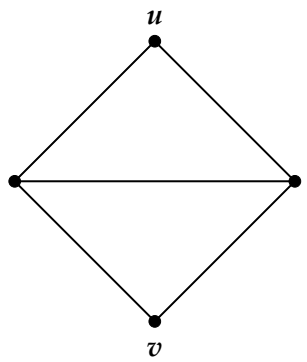


Figure 4.3: The graph $K_4 - uv$.

Lemma 4.2.2 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that $|E| = 2|V| - 3$, E is independent in $\mathcal{M}_{uv}(G)$, and $d(a) \geq 3$ for all $a \in V - \{u, v\}$. Then either $G = K_4 - uv$ or there is a vertex $z \in V - \{u, v\}$ with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$.*

Proof. For a contradiction suppose that for all $z \in V - \{u, v\}$ with $d(z) = 3$ we have $z \in N(u) \cap N(v)$ and let m denote the number of vertices of degree three in $N(u) \cap N(v)$. We may assume that $m \leq d(u) \leq d(v)$. By our assumptions we have

$$4|V| - 6 = 2|E| = \sum d(v) \geq d(u) + d(v) + 3m + 4(|V| - m - 2)$$

$$= 4|V| - m + d(u) + d(v) - 8 \geq 4|V| + d(v) - 8,$$

which implies that $m = d(u) = d(v) = 2$ must hold. Let $N(u) \cap N(v) = \{a, b\}$. Then either $ab \in E$ and hence $G = K_4 - uv$ or $U = V - \{u, v, a, b\}$ is non-empty and $i(U) \geq 2|U| - 1$ holds, contradicting the fact that E is independent in $\mathcal{M}_{uv}(G)$. $\square$

Lemma 4.2.3 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that E is independent in $\mathcal{M}_{uv}(G)$ and let $z \in V - \{u, v\}$ be a vertex with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$. Then there is a 1-reduction at z which leads to a graph G' which is independent in $\mathcal{M}_{uv}(G')$.*

Proof. Let $F = \{ab \notin E : a, b \in N(z)\}$, let $G_1 = G - z + F$ and $G_2 = G + F$. For a contradiction suppose that $r_{uv}(G_1) \leq r_{uv}(G) - 3$. Consider a base B_1 of $\mathcal{M}_{uv}(G_1)$ which contains the triangle on $N(z)$ and let B_2 be a base of $\mathcal{M}_{uv}(G_2)$ with $B_1 \subseteq B_2$. Since K_4 is a circuit of $\mathcal{M}_{uv}(G_2)$, we have $r_{uv}(G_2) \leq r_{uv}(G_1) + 2$. Thus $r_{uv}(G) \leq r_{uv}(G_2) \leq r_{uv}(G) - 1$, a contradiction. □

Theorem 4.2.4 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-independent if and only if E is independent in $\mathcal{M}_{uv}(G)$.*

Proof. Necessity follows from Lemma 4.1.7. Now suppose that E is independent in $\mathcal{M}_{uv}(G)$. We prove that G is uv-independent by induction on $|V|$. By extending E to a base of $\mathcal{M}_{uv}(K_{|V|})$, if necessary, we may assume that $|E| = 2|V| - 3$ holds. If $|V| \leq 4$ then we must have $G = K_4 - uv$, which is uv-independent. Thus we may assume that $|V| \geq 5$.

First suppose that there is a vertex $w \in V - \{u, v\}$ with $d(w) = 2$. Let $N(w) = \{a, b\}$. Clearly, $a \neq b$ holds. If $\{a, b\} = \{u, v\}$ then let $\mathcal{H} = \{\{u, v, w\}, \{V - w\}\}$. We have

$$2|V| - 3 = |E| = i_E(\mathcal{H}) \leq \mathrm{val}(\mathcal{H}) = 2 \cdot 3 - 3 + 2(|V| - 1) - 3 - 2 = 2|V| - 4,$$

a contradiction. Hence $\{a, b\} \neq \{u, v\}$, which implies that the 0-uv-reduction operation can be applied at w to obtain a graph $G' = (V - w, E')$ that is independent in the matroid $\mathcal{M}_{uv}(G')$ and satisfies $|E'| = 2|V - w| - 3$. By induction, G' is uv-independent. Now Lemma 4.2.1 implies that G is uv-independent.

Next suppose that there is no vertex of degree two in G. By Lemmas 4.2.2 and 4.2.3 we may apply the 1-uv-reduction operation at some vertex z of degree three to obtain a graph $G' = (V - w, E')$ that is independent in the matroid $\mathcal{M}_{uv}(G')$ and satisfies $|E'| = 2|V - w| - 3$. By induction G' is uv-independent. Lemma 4.2.1 implies that G is uv-independent. This completes the proof. □

As a by-product of the proof of Theorem 4.2.4 we obtain the following corollary.

Theorem 4.2.5 *Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$ and let $u, v \in V$ be distinct vertices. Then G is uv-independent if and only if G can be obtained from $K_4 - uv$ by a sequence of 0-uv-extensions and 1-uv-extensions.*

4.2.1 Main result

Theorem 4.2.6 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-rigid if and only if $G - uv$ and G_{uv} are both rigid.*

Proof. Necessity follows from the fact that an infinitesimally rigid uv-coincident realization of G gives rise to an infinitesimally rigid realization of $G - uv$ as well as G_{uv}, by (4.2).

To prove sufficiency, suppose, for a contradiction, that $G - uv$ and G_{uv} are both rigid but G is not uv-rigid. By Theorems 6.6.5 and 4.2.4 this implies that there is a thin cover $\mathcal{K}$ of $G - uv$ with $\mathrm{val}(\mathcal{K}) \leq 2|V| - 4$. If $\mathcal{K}$ consists of subsets of V only, then $r(G - uv) \leq 2|V| - 4$ follows, which contradicts the fact that $G - uv$ is rigid.

Hence $\mathcal{K} = \{\mathcal{H}, H_1, \ldots, H_k\}$, where $\mathcal{H} = \{X_1, \ldots, X_l\}$ is a uv-compatible family. Contract the vertex pair u, v in G into a new vertex z_{uv}. This leads to a graph G_{uv} and a cover

$$\mathcal{K}' = \{X_1', \ldots, X_l', H_1, \ldots, H_k\}$$

of G_{uv}, where X_j' is obtained from X_j by replacing u, v by z_{uv}, for $1 \leq j \leq l$. Then we obtain

$$\sum_{i=1}^{k}(2|H_i| - 3) + \sum_{j=1}^{l}(2|X_j'| - 3) = \sum_{i=1}^{k}(2|H_i| - 3) +$$

$$+ \sum_{j=1}^{l}(2|X_j| - 3) - 2l = \mathrm{val}(\mathcal{K}) - 2 \leq 2|V| - 4 - 2 = 2(|V| - 1) - 4,$$

which implies that G_{uv} is not rigid, a contradiction. This completes the proof. □

A similar proof can be used to verify the following more general result:

Theorem 4.2.7 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then $r_{uv}(G) = \min\{r(G - uv), r(G_{uv}) + 2\}$.*

Theorems 4.2.6 and 4.2.7 show that the polynomial-time algorithms for computing the rank of a graph in the two-dimensional rigidity matroid (see e.g. [7]) can be used to test whether G is uv-rigid, or more generally, to compute $r_{uv}(G)$.

4.3 An obstacle for minimal uv-rigidity

We may also obtain a characterization of minimally uv-rigid graphs which is similar to the obstacle-based characterization for the collinear problem given in Theorem 4.0.2.

Theorem 4.3.1 *Let $G = (V, E)$ be a minimally rigid graph and let $u, v \in V$ be distinct vertices. Suppose that $uv \notin E$. Then the following statements are equivalent:*
(i) G is uv-rigid,
(ii) there is no subgraph $G' = (V', E')$ of G with $\{u, v\} \subseteq V'$ and $|E'| = 2|V'| - (3+s)$ such that $G' - \{u, v\}$ has at least $s + 2$ components, for $s = 0$ or $s = 1$.

Proof. First suppose that there is a subgraph $G' = (V', E')$ of G with $|E'| = 2|V'| - (3+s)$ for which $G' - \{u, v\}$ has at least $s+2$ components, for $s = 0$ or $s = 1$. Let $G_1 = (E_1, V_1), \ldots, G_t = (E_t, V_t)$ be the components of $G - \{u, v\}$. Consider the following cover of G:

$$\mathcal{K} = \{\{V_i \cup \{u, v\} : 1 \leq i \leq t\}\} \cup \{\{v_p, v_q\} : v_p v_q \in E - E'\}.$$

Since $t \geq s + 2$, we obtain

$$r_{uv}(E) \leq \sum_{i=1}^{t} (2|V_i + \{u, v\}| - 3) - 2(t-1) + |E - E'| = \sum_{i=1}^{t} 2|V_i| - t + 2 + |E - E'| =$$

$$= 2|(\bigcup_{i=1}^{t} V_i) \cup \{u, v\}| - (t+2) + |E - E'| \leq 2|V'| - (s+4) + |E - E'| < |E|.$$

Thus G is not uv-independent (and hence not uv-rigid) by Lemma 4.1.7. Hence (i) implies (ii).

Next suppose that G is not uv-rigid. Then, by Theorems 6.6.5 and 4.2.4, there is a thin cover $\mathcal{K}_0$ of G with $\mathrm{val}(\mathcal{K}_0) \leq 2|V| - 4$. Since G is rigid, $\mathcal{K}_0 = \{\mathcal{H}, H_1, \ldots, H_k\}$, where $\mathcal{H} = \{X_1, \ldots, X_l\}$ is a uv-compatible family with $l \geq 2$. Since $\mathcal{K}_0$ is thin, the set $\{u, v\}$ separates the subgraph $G' = (V', E')$, where $V' = V(\mathcal{H})$ and $E' = E(\mathcal{H}) = E(V')$.

We claim that by choosing $\mathcal{K}_0$ so that the number of its members is maximized, we have $i(H_i) = 2|H_i| - 3$ for all $1 \leq i \leq k$ and $i(X_i) \geq 2|X_i| - 4$ for all $1 \leq j \leq l$. The claim follows by observing that we can replace a set H_i or X_j violating these counts by the pairs of end-vertices of the edges it covers to obtain another cover with the same or smaller value. (If $X_j \in \mathcal{H}$ then we also remove X_j from the uv-compatible family.) Furthermore, since G is independent and $uv \notin E$, there can be at most one $X_i \in \mathcal{H}$ with $E(X_i) = 2|X_i| - 3$, c.f. Lemma 4.1.4.

If there is a $X_i \in \mathcal{H}$ with $E(X_i) = 2|X_i| - 3$ then it is easy to see that we have $|E'| = 2|V'| - 3$. Since $l \geq 2$, $G' - \{u, v\}$ has at least two components.

If $E(X_i) = 2|X_i| - 4$ for all $1 \leq i \leq l$ then we have $|E'| = 2|V'| - 4$ and $l \geq 3$. To see the latter inequality suppose that $l = 2$ and take the cover $\mathcal{K}_3 = \{H_1, \ldots, H_k\} \cup \{\{n_a, n_b\} : n_a n_b \in E(X_1)\} \cup \{\{n_a, n_b\} : n_a n_b \in E(X_2)\}$. We have $\mathrm{val}(\mathcal{K}_3) = \mathrm{val}(\mathcal{K}_0) < 2|V| - 3$. Since there is no uv-compatible family in $\mathcal{K}_3$, this

contradicts the fact that G is rigid. Hence $l \geq 3$, as claimed, which implies that $G' - \{u, v\}$ has at least three components. Thus (ii) implies (i). $\square$

Finally we remark that the counterpart of [22, Corollary 4.4] is not true for uv-coincident realizations. It is easy to find a minimally rigid graph G with an arbitrary large number of vertices such that for a fixed $v \in V(G)$ there is no $u \in V(G)$ for which G is uv-rigid. (Let G and v be such that v is connected with every other vertex of G.) Furthermore, there is no pair u, v in $K_{3,3}$ for which $K_{3,3}$ would be uv-rigid (an example due to John Owen).

Chapter 5

Sparse hypergraphs with applications in combinatorial rigidity

In this chapter we develop a new inductive construction of 4-regular $(1,3)$-tight hypergraphs and use it to solve problems in combinatorial rigidity.

We give a combinatorial characterization of generically projectively rigid hypergraphs on the projective line. Our result also implies an inductive construction of generically minimally affinely rigid hypergraphs in the plane. Based on the rank function of the corresponding count matroid on the edge set of H we obtain combinatorial proofs for some sufficient conditions for the generic affine rigidity of hypergraphs.

5.1 Introduction

Our goal is to provide combinatorial tools for attacking problems from rigidity theory in which the underlying combinatorial structure is a hypergraph: projective rigidity, affine rigidity, and scene analysis.

We develop a new inductive construction of 4-regular $(1,3)$-tight hypergraphs. By using this result we give a combinatorial characterization of generically projectively rigid hypergraphs on the projective line, which was conjectured by George and Ahmed [14]. Our result also implies an inductive construction of generically minimally affinely rigid hypergraphs in the plane. Based on the rank function of the corresponding count matroid on the edge set of H we obtain combinatorial proofs for some sufficient conditions for the generic affine rigidity of hypergraphs, due to Gortler, Gotsman, Liu, and Thurston [16] and Zha and Zhang [74], respectively.

5.2 Inductive constructions

Let $H = (V, E)$ be a hypergraph and let $X \subseteq V$. We use $i_H(X)$ to denote the number of edges induced by X in H. The hypergraph is called *m-uniform*, for some positive integer m, if each hyperedge $e \in E$ contains exactly m vertices. The degree of a vertex v in H is denoted by $d_H(v)$ and the number of edges of H that contain a given pair $v, w \in V$ is denoted by $d_H(v, w)$. We may omit the subscript referring to H if it is clear from the context.

We introduce a set of operations on $(k+1)$-uniform hypergraphs which preserve $(1, k)$-sparsity and which can be used to generate all $(k + 1)$-uniform $(1, k)$-tight hypergraphs from a single hyperedge, for all $1 \leq k \leq 3$.

Let $H = (V, E)$ be a $(k + 1)$-uniform hypergraph, let j be an integer with $0 \leq j \leq k - 1$, and let $v \in V$ be a vertex with $d(v) \geq j$. The *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size $k + 1$ incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$. Thus the new vertex z has degree $j + 1$ in the extended hypergraph. See Figure **??**. Note that a 0-extension operation simply adds a new vertex z and a new hyperedge of size $k + 1$ incident with z.

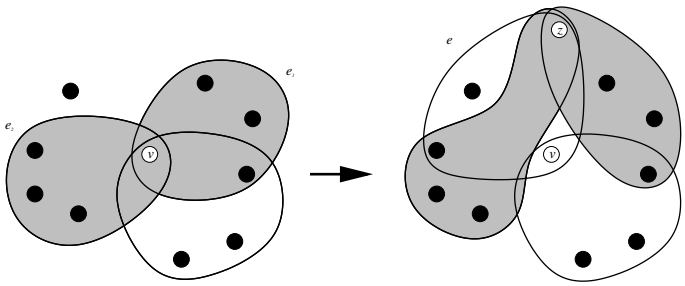


Figure 5.1: A 2-extension operation on a 4-uniform hypergraph.

The j-extension operation preserves sparsity in the following sense. The simple proof of the next lemma is omitted.

Lemma 5.2.1 *Let $H = (V, E)$ be a $(k+1)$-uniform $(1, k)$-sparse ($(1, k)$-tight) hypergraph and let H' be obtained from H by a j-extension operation, where $0 \leq j \leq k-1$. Then H' is also $(1, k)$-sparse ($(1, k)$-tight, respectively).*

The inverse operation of j-extension can be defined as follows. Let $H = (V, E)$ be a $(k + 1)$-uniform hypergraph. Consider a vertex $z \in V$ with $d(z) = j + 1$, for some $0 \leq j \leq k - 1$, and let v be a neighbour of z in H with $d(z, v) = 1$. Let $e_1, e_2, ..., e_{j+1}$ be the edges incident with z, where e_1 is the unique edge which is incident with v,

too. The *j-reduction operation at vertex z on neighbour v* deletes e_1 and replaces e_i by $e_i - z + v$ for all $2 \leq i \leq j+1$. Observe that the inverse of j-extension is indeed j-reduction.

We say that a j-reduction operation in a $(k+1)$-uniform $(1,k)$-sparse hypergraph H is *admissible* if the hypergraph obtained from H by the operation is also $(1,k)$-sparse. To obtain our inductive construction by induction we shall show that each $(k+1)$-uniform $(1,k)$-sparse hypergraph H (for k up to 3) has a vertex z of degree at most k and that there exists an admissible $(d(z)-1)$-reduction at z.

Note that the 2-uniform $(1,1)$-tight hypergraphs are the trees, for which the existence of a vertex of degree one (a leaf) and an admissible 0-reduction (leaf deletion) is straightforward. The case when $k = 2$ is more complicated, but still not very difficult, so we shall omit the proof of this case. Instead, we shall focus on 4-regular $(1,3)$-tight hypergraphs (see Theorem 5.2.8 below, which is the main result of this section).

It should also be noted that the above proof strategy does not work when $k \geq 4$. To see this consider the $(1,4)$-tight 5-uniform hypergraph $H = (V,E)$ with $V = \{v_1, v_2, ..., v_7\}$ and $E = \{(v_1, v_2, v_3, v_4, v_7), (v_3, v_4, v_5, v_6, v_7), (v_1, v_2, v_5, v_6, v_7)\}$. We have $d(v_7) = 3$ but each neighbour v_i of v_7 has $d(v_7, v_i) \geq 2$, showing that no 2-reduction can be performed at v_7. Hence an inductive construction for higher values of k is probably more difficult to obtain.

Before dealing with the case of $(1,3)$-sparse hypergraphs we prove some preliminary lemmas about $(1,k)$-sparse hypergraphs in general. The next lemma is easy to verify by observing that the contribution of a hyperedge to the right hand side of inequality (5.1) below cannot be less than its contribution to the left hand side.

Lemma 5.2.2 *Let $H = (V,E)$ be a hypergraph and let $X, Y \subseteq V$ be subsets of vertices. Then*

$$i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y). \tag{5.1}$$

Let $H = (V,E)$ be a $(k+1)$-uniform $(1,k)$-sparse hypergraph. We say that a subset $X \subseteq V$ is *critical* if $i(X) = |X| - k$ holds. A subset $Y \subseteq V$ is called *semi-critical* if $i(Y) \geq |Y| - k - 1$.

Lemma 5.2.3 *Let $H = (V,E)$ be a $(k+1)$-uniform $(1,k)$-sparse hypergraph and let $X, Y \subseteq V$ be subsets of vertices. If $|X \cap Y| \geq k$ and*
(i) if X and Y are both critical then $X \cup Y$ is also critical,
(ii) if X is critical and Y is semi-critical then $X \cup Y$ is semi-critical,
(iii) if X and Y are both semi-critical and $X \cap Y$ is not critical then $X \cup Y$ is semi-critical.

Furthermore,
(iv) if $|X \cap Y| = k-1$ and X and Y are both critical then $X \cup Y$ is semi-critical.

Proof. Suppose that X and Y are both critical. Then, by using Lemma 5.2.2, we can deduce that

$$|X| - k + |Y| - k = i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y) \leq$$

$$\leq |X \cup Y| - k + |X \cap Y| - k = |X| - k + |Y| - k.$$

Thus we must have equality everywhere, which implies that $X \cup Y$ is also critical. This proves (i). The proofs of (ii), (iii), and (iv) are similar. □

We also need the following observation.

Lemma 5.2.4 *Let $H = (V, E)$ be a $(k+1)$-uniform $(1, k)$-tight hypergraph with $|V| \geq k+1$. Then*
(i) $d(v) \geq 1$ for all $v \in V$, and
(ii) there is a vertex $z \in V$ with $d(z) \leq k$.

In what follows we shall consider the case when $k = 3$ and H is tight, that is, when H is a 4-uniform $(1, 3)$-tight hypergraph. Let $H = (V, E)$ be a hypergraph, $z \in V$, and $X \subseteq V$. We denote the set of neighbours of z in H by $N_H(z)$ and the number of edges e of H with $z \in e$ and $e \subseteq X \cup \{z\}$ by $e(z, X)$.

Theorem 5.2.5 *Let $H = (V, E)$ be a $(1, 3)$-tight 4-uniform hypergraph and let $z \in V$ be a vertex with $d(z) = j+1$ for some $0 \leq j \leq 2$. Then there is an admissible j-reduction at z.*

Proof. First suppose that $d(z) = 1$. Then the 0-reduction at z, which deletes the unique edge incident with z, is clearly admissible. Next suppose that $d(z) = 2$ holds and let e_1, e_2 be the hyperedges incident with z. The following property, which is implied by the sparsity of H, will be used several times in the proof. Let X be a subset of $V - z$. Then

(*) if X is critical then $e(z, X) \leq 1$ and if X is semi-critical then $e(z, X) \leq 2$ holds.

To show the existence of an admissible 1-reduction at z we have to show that for some neighbour v of z, for which $d(z, v) = 1$, the hypergraph obtained from H by deleting z and adding $e_2 - z + v$ is $(1, 3)$-sparse, where e_1 is the unique edge containing z and v. Observe that the addition of the new hyperedge $e_2 - z + v$ destroys $(1, 3)$-sparsity if and only if there is a critical set $X \subseteq V - z$ with $e_2 - z + v \subseteq X$.

Since H is $(1,3)$-sparse and $d(z) = 2$, we have $4 \leq |N(z)| \leq 6$. Hence there exists a vertex $a \in N(z)$ with $d(z,a) = 1$. Let $e_1 = (a,b,c,z)$ and $e_2 = (d,e,f,z)$. If $|N(z)| = 4$ then the 2-reduction at z on neighbour a is admissible, for otherwise there exists a critical set X with $N(z) = \{a,d,e,f\} \subseteq X$ and $e(z,X) \geq 2$, contradicting (*). If $|N(z)| = 5$ then we may assume that $c = d$ and $e_1 - e_2 = \{a,b\}$. Hence $d(z,a) = d(z,b) = 1$. By assuming that the 1-reductions at a and b are both non-admissible we could deduce that there exist critical sets X, Y with $\{a,d,e,f\} \subseteq X$ and $\{b,d,e,f\} \subseteq Y$. Then, by Lemma 5.2.3(i), $X \cup Y$ would also be critical. Since $N(z) \subseteq X \cup Y$, this would again contradict (*). The case when $|N(z)| = 6$ is similar. Thus there is an admissible 1-reduction at z.

The last case to consider is when $d(z) = 3$. Let N_1 denote the set of neighbours x of z with $d(z,x) = 1$. Notice that

$$9 = \sum_{x \in N(z)} d(z,x) \geq 2|N(z) - N_1| + |N_1| = 2|N(z)| - |N_1|, \tag{5.2}$$

so $|N_1| \geq 2|N(z)| - 9$. Since H is $(1,3)$-sparse and $d(z) = 3$, we have $5 \leq |N(z)| \leq 9$. Hence $N_1 \neq \emptyset$.

Let e_1, e_2, e_3 be the hyperedges incident with z. To show the existence of an admissible 2-reduction at z we have to show that for some neighbour v of z, for which $d(z,v) = 1$, the hypergraph obtained from H by deleting z and adding $e_2 - z + v$ and $e_3 - z + v$ is $(1,3)$-sparse, where e_1 is the unique edge containing z and v. Observe that the addition of the new hyperedges $e_2 - z + v$ and $e_3 - z + v$ destroys $(1,3)$-sparsity if and only if there is a critical set $X \subseteq V - z$ with $e_i - z + v \subseteq X$, for some $2 \leq i \leq 3$, or there is a semi-critical set $Y \subseteq V - z$ with $(e_2 - z + v) \cup (e_3 - z + v) \subseteq Y$. These critical or semi-critical sets X or Y, which show that the 3-reduction at z with v is non-admissible, are called the *blockers* of v.

For a contradiction suppose that there is no admissible 3-reduction at z. Then each vertex in N_1 has a blocker.

Claim 5.2.6 *Each blocker is critical.*

Proof. Let $x \in N_1$ and suppose, for a contradiction, that x has a blocker Y which is not critical. Let (z,x,a,b) be the unique edge containing z and x. Thus Y contains all neighbours of z, except, possibly, a and b. We may suppose that Y is a maximal semi-critical blocker of x. If $a, b \in Y$ then $N(z) \subseteq Y$ and $e(z,Y) \geq 3$ follow, contradicting (*). If, say, $a \notin Y$ then $a \in N_1$. Consider a blocker X of a. If X is critical then $X \cup Y$ is also a semi-critical blocker of x by Lemma 5.2.3(ii), contradicting the maximality of Y. If X is semi-critical then X contains all neighbours of z, except, possibly, x and b. Since $e(z, X \cap Y) \geq 2$, (*) implies that $X \cap Y$ is not critical. By using Lemma

5.2.3(iii) we conclude that $X \cup Y$ is semi-critical, contradicting the maximality of Y. $\square$

Claim 5.2.7 *Every edge e incident with z contains a vertex w with $d(z,w) \geq 2$.*

Proof. For a contradiction suppose, without loss of generality, that $(e_1 - z) \cap (e_2 \cup e_3) = \emptyset$. Let $e_1 = (a,b,c,z)$. Then $a,b,c \in N_1$ and by the sparsity of H we also have $e_2 \cap N_1 \neq \emptyset$ and $e_3 \cap N_1 \neq \emptyset$. By symmetry may suppose that $e_2 - z \subseteq X_a \cap X_b$, where X_a and X_b are critical blockers of a and b, respectively. By Lemma 5.2.3(i) $X_a \cup X_b$ is also critical.

Let $f \in e_2 \cap N_1$ and let Z be a critical blocker of f. If $(e_1 - z) \subseteq Z$ then, by Lemma 5.2.3(i), $X_a \cup X_b \cup Z$ is also critical. Since $e(z, X_a \cup X_b \cup Z) \geq 2$, this contradicts (*). So we may suppose that the critical blocker Z_i of each vertex $f_i \in e_2 \cap N_1$, $1 \leq i \leq |e_2 \cap N_1|$, satisfies $(e_3 - z) \subseteq Z_i$. But, again by Lemma 5.2.3(i), this would imply that the union Z' of these sets Z_i is also critical. Since $e(z, Z') \geq 2$, this would contradict (*). This proves the claim. $\square$

Claim 5.2.7 implies that $|N(z)| \leq 7$. First suppose that $|N(z)| = 5$. Let X be a critical blocker of some vertex $x \in N_1$. Then $|N(z) - X| \leq 1$ and hence $Y = N(z) \cup X$ is semi-critical. Since $e(z, Y) = 3$, this contradicts (*).

Next suppose that $|N(z)| = 6$. Then we have $|N_1| \geq 3$ by (5.2). Hence we can find two critical blockers X_a, X_b belonging to two distinct vertices $a, b \in N_1$. Each of these blockers contains at least four neighbours of z. If $N(z) \subseteq (X_a \cup X_b)$ then $|X_a \cap X_b| \geq 2$. Thus, by Lemma 5.2.3(i),(iv), it follows that $X_a \cup X_b$ is semi-critical. Since $e(z, X_a \cup X_b) = 3$, this contradicts (*). If $|(X_a \cup X_b) \cap N(z)| = 5$ then a similar argument, using Lemma 5.2.3(i) gives that $X_a \cup X_b$ is critical and hence $Y = X_a \cup X_b \cup N(z)$ is semi-critical, contradicting (*). If $|(X_a \cup X_b) \cap N(z)| = 4$ then $X_a \cup X_b$ is critical, with $e(z, X_a \cup X_b) = 2$, contradicting (*).

It remains to consider the case when $|N(v)| = 7$. First suppose that there is a vertex $w \in N(z)$ with $d(z,w) = 3$. Then $N_1 = N(z) - w$ must hold. By symmetry we may suppose that for some vertex $a \in e_1 \cap N_1$ and a critical blocker X_a of a we have $(e_2 - z) \subset X_a$. Let $e_2 = (z,w,c,d)$. Let X_c, X_d be critical blockers of c and d, respectively. If $(e_1 - z) \subset X_c$ then $X_a \cup X_c$ is critical, by Lemma 5.2.3(i), and has $e(z, X_a \cup X_c) \geq 2$, contradicting (*). A similar argument works for X_d. So we may assume that $(e_3 - z) \subseteq X_c \cap X_d$. But then, by Lemma 5.2.3(i), $X_c \cup X_d$ is a critical set with $e(z, X_c \cup X_d) \geq 2$, contradicting (*).

Next suppose that each vertex $w \in N(z)$ has $d(z,w) \leq 2$. Then we have two vertices $p, q \in N(z)$ with $d(z,p) = d(z,q) = 2$ and the other neighbours of z are all in N_1. Furthermore, by using Claim 5.2.7, we can deduce that the edges incident

with z can be labeled as $e_1 = (p, a, b, z)$, $e_2 = (p, q, c, z)$, and $e_3 = (q, d, e, z)$. By symmetry we may suppose that a critical blocker X_c of c has $(e_1 - z) \subset X_c$. Let X_a and X_b be critical blockers of a and b, respectively. If, say, $(e_2 - z) \subset X_a$ holds then, by Lemma 5.2.3(i) it follows that $X_a \cup X_c$ is critical. Since $e(z, X_a \cup X_c) \geq 2$, this contradicts (*). A similar argument works for X_b. Thus we may assume that $(e_3 - z) \subseteq X_a \cap X_b$. This gives that $X_a \cup X_b$ is critical and $Y = X_a \cup X_b \cup X_c$ is semi-critical, by using Lemma 5.2.3(i) and (iv), respectively. Since $e(z, Y) = 3$, this contradicts (*). With this final contradiction the proof of the theorem is complete. □

As a corollary we obtain the main result of this section.

Theorem 5.2.8 *Let $H = (V, E)$ be a 4-uniform hypergraph. H is $(1, 3)$-tight if and only if it can be obtained from a single hyperedge of size four by a sequence of 0-extensions, 1-extensions, and 2-extensions.*

Proof. The 'if' part follows from Lemma 5.2.1. Theorem 5.2.5 implies the 'only if' part by induction on the number of vertices. □

One can prove a similar result about reductions in 3-uniform $(1, 2)$-tight hypergaphs, which leads to the following inductive construction. The proof, which is similar to the first part of the proof of Theorem 5.2.5, where $d(z) \leq 2$, is omitted.

Theorem 5.2.9 *Let $H = (V, E)$ be a 3-uniform hypergraph. H is $(1, 2)$-tight if and only if it can be obtained from a single hyperedge of size three by a sequence of 0-extensions and 1-extensions.*

5.3 Projective rigidity on the line

In a recent manuscript George and Ahmed [14] initiated the study of rigidity properties of projective frameworks. A one-dimensional *projective framework* (H, p) is a pair, where H is a 4-uniform hypergraph and p is a map from $V(H)$ to distinct points of the one-dimensional projective space $\mathbb{P}^1$. They call a smooth deformation of the framework a flex if it preserves the cross ratio[1] for each 4-tuple that belongs to the edge set of H and call a framework rigid if it has only trivial flexes (that is, restrictions of a combination of some translation, scaling, and rotation of the

[1]Recall that the *cross ratio* of four points a, b, c, d, in this order, is

$$R(ab, cd) = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

space). As in the case of bar-and-joint frameworks with length constraints, one may define the infinitesimal rigidity of projective frameworks by considering the rank of the following *projective rigidity matrix* $Q(H,p)$ of framework (H,p), in which the entries are obtained as partial derivatives of a smooth flex at time zero. Let us fix an ordering $(v_1, v_2, ..., v_n)$ of $V(H)$ and define $Q(H,p)$ to be the $|E| \times |V|$ matrix in which each row (resp. column) corresponds to an edge (resp. vertex) of H. The row corresponding to some edge $e = \{v_i, v_j, v_k, v_l\}$ with $i < j < k < l$ has non-zero entries only in the columns of v_i, v_j, v_k, v_l, and has the following form:

$$\begin{pmatrix} 0...0 & \frac{(b-d)(c-d)}{(b-c)(a-d)^2} & 0...0 & \frac{(a-c)(d-c)}{(a-d)(b-c)^2} & 0...0 & \frac{(d-b)(b-a)}{(a-d)(b-c)^2} & 0...0 & \frac{(c-a)(a-b)}{(b-c)(a-d)^2} & 0...0 \end{pmatrix}$$

where we put $p(v_i) = a, p(v_j) = b, p(v_k) = c, p(v_l) = d$ for simplicity.

It is not hard to show that the rank of $Q(H,p)$ cannot exceed $|V(H)| - 3$. A realization (H,p) of a 4-uniform hypergraph $H = (V,E)$ in $\mathbb{P}^1$ is *infinitesimally projectively rigid* if $\operatorname{rank} Q(H,p) = |V| - 3$ (for some ordering of $V(H)$). We say that $H = (V,E)$ is *generically projectively rigid* in $\mathbb{P}^1$ if there exists an infinitesimally projectively rigid realization of H in $\mathbb{P}^1$. A *minimally generically projectively rigid* hypergraph is a projectively rigid hypergraph with $|E| = |V| - 3$. Note that the entries of the matrix depend on the chosen ordering of $V(H)$ in a non-trivial way, just like the cross ratio. We shall prove that the rank of the matrix does not depend on the ordering (assuming that it equals $|V| - 3$ for some ordering) and hence the chosen ordering of $V(H)$ does not matter. It should also be noted that a hypergraph H is generically projectively rigid if and only if every *generic* framework (H,p) is infinitesimally projectively rigid.

George and Ahmed [14] showed that an infinitesimally rigid projective framework is rigid. They also pointed out that a minimally generically projectively rigid hypergraph is $(1,3)$-tight and conjectured that this sparsity condition is also sufficient to guarantee minimal projective rigidity. As an application of our inductive construction (Theorem 5.2.8) in the rest of this section we shall prove this conjecture.

Lemma 5.3.1 *Let (H,p) be a one-dimensional projective framework on n vertices. Suppose that* $\operatorname{rank} Q(H,p) = n - 3$. *Then any set of $n-3$ columns of $Q(H,p)$ is linearly independent.*

Proof. First observe that the kernel of $Q(H,p)$ is at least three-dimensional, as it contains the linearly independent vectors $\mathbf{1} = (1,1,\ldots,1)$, $\mathbf{p} = (p(v_1), p(v_2), \ldots, p(v_n))$, and $\mathbf{p}^2 = (p(v_1)^2, p(v_2)^2, \ldots, p(v_n)^2)$. Let C_i denote the column of $Q(H,p)$ that corresponds to vertex v_i, $1 \le i \le n$. Let us fix a triple $\{j,k,l\} \subseteq \{1,\ldots,n\}$. We shall prove that C_l is spanned by the set of columns $\{C_t : 1 \le t \le n, t \ne j,k,l\}$, from

which the lemma follows. Let $x = -p(v_j) - p(v_k)$ and $y = p(v_j)p(v_k)$. Then we have $p(v_j)^2 + xp(v_j) + y = p(v_k)^2 + xp(v_k) + y = 0$. Furthermore, $p(v_t)^2 + xp(v_t) + y = 0$ if and only if $t \in \{j, k\}$.

Consider the vector $\mathbf{p}^2 + x\mathbf{p} + y\mathbf{1}$, which is in the kernel of $Q(H, p)$. This gives rise to a linear combination of the columns of $Q(H, p)$, which gives the zero vector, and in which the coefficients of C_j, C_k are zeros and the coefficient of C_l is nonzero. Thus C_l is spanned by the set of columns $\{C_t : 1 \leq t \leq n, t \neq j, k, l\}$, as claimed. □

The next lemma implies that if the rank of the projective rigidity matrix attains $|V(H)| - 3$ for some ordering of $V(H)$, then it is the same for all orderings.

Lemma 5.3.2 *Let (H, p) be a one-dimensional projective framework on n vertices. Let $Q(H, p)$ be the projective rigidity matrix in which the columns are labeled by vertices $v_1, v_2, \ldots, v_n$, in this order. Suppose that* $\operatorname{rank} Q(H, p) = n - 3$. *Let $Q'(H, p)$ be the projective rigidity matrix of (H, p) corresponding to the labeling $v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2} \ldots, v_n$ for some $1 \leq i \leq n - 1$. Then* $\operatorname{rank} Q'(H, p) = n - 3$.

Proof. Let $Q_1(H, p)$ and $Q'_1(H, p)$ be the matrices that we get by deleting the columns of v_i and v_{i+i} from $Q(H, p)$ and $Q'(H, p)$, respectively. By Lemma 5.3.1 $\operatorname{rank} Q_1(H, p) = n - 3$.

We will show that every row of $Q'_1(H, p)$ can be obtained from the corresponding row of $Q_1(H, p)$ by multiplying it with an appropriate scalar. First observe that if hyperedge e contains at most one of v_i and v_{i+i} then the rows corresponding to e in $Q_1(H, p)$ and $Q'_1(H, p)$ are equal. Suppose that $e = v_j v_k v_i v_{i+1}$. We split the proof into three cases.

First suppose that $j < k < i$. Put $p(v_j) = a, p(v_k) = b, p(v_i) = c, p(v_{i+1}) = d$. With this notation the two nonzero entries of the row of e in $Q_1(H, p)$ are:

$$\frac{(b-d)(c-d)}{(b-c)(a-d)^2}, \frac{(a-c)(d-c)}{(a-d)(b-c)^2};$$

while the two entries in $Q'_1(H, p)$ are:

$$\frac{(b-c)(d-c)}{(b-d)(a-c)^2}, \frac{(a-d)(c-d)}{(a-c)(b-d)^2}.$$

Hence we can get the row of e in $Q'_1(H, p)$ by multiplying the corresponding row in $Q_1(H, p)$ with the scalar $-\frac{(b-c)^2(a-d)^2}{(b-d)^2(a-c)^2}$.

If $j < i < k$ then let $p(v_j) = a, p(v_i) = b, p(v_{i+1}) = c, p(v_k) = d$. Now the the two nonzero entries of the row of e in $Q_1(H, p)$ are:

$$\frac{(b-d)(c-d)}{(b-c)(a-d)^2}, \frac{(c-a)(a-b)}{(b-c)(a-d)^2}$$

and we get the corresponding row of $Q_1'(H,p)$ by multiplying with -1.

The last case is when $i < j < k$. Now put $p(v_i) = a, p(v_{i+1}) = b, p(v_j) = c, p(v_k) = d$. Here the two nonzero entries of the rows are

$$\frac{(d-b)(b-a)}{(a-d)(b-c)^2}, \frac{(c-a)(a-b)}{(b-c)(a-d)^2};$$

and

$$\frac{(a-d)(b-a)}{(b-d)(a-c)^2}, \frac{(b-c)(a-b)}{(a-c)(b-d)^2}.$$

In this case the appropriate scalar is $-\frac{(a-d)^2(b-c)^2}{(d-b)^2(a-c)^2}$.

Thus $\operatorname{rank} Q'(H,p) \geq \operatorname{rank} Q_1'(H,p) \geq \operatorname{rank} Q_1(H,p) = n-3$, as required. $\square$

We are ready to prove the main result of this section.

Theorem 5.3.3 *Let $H = (V,E)$ be a 4-uniform hypergraph. Then H is minimally generically projectively rigid in $\mathbb{P}^1$ if and only if H is $(1,3)$-tight.*

Proof. We have to show that there exists a realization (H,p) of H in $\mathbb{P}^1$ with $\operatorname{rank} Q(H,p) = |V| - 3$. We prove this by induction on $|V|$. If $|V| = 4$ then any realization will do. Now suppose that $|V| \geq 5$ and the theorem holds for all 4-uniform hypergraphs with at most $|V| - 1$ vertices. By Theorem 5.2.8 H can be obtained from a 4-uniform hypergraph $H' = (V', E')$ by a j-extension operation at some vertex $v \in V'$, where $0 \leq j \leq 2$. Recall that the operation adds a new vertex z, replaces v by z in j edges incident with v and adds an additional edge e incident with v and z. Let F be the (possibly empty) set of the j edges replaced.

By induction, H' has a realization (H', p') for which $\operatorname{rank} Q(H', p') = |V'| - 3 = |V| - 4$ holds. We may suppose that the last j rows of $Q(H', p')$ correspond to the edges in F. By Lemma 5.3.2 we may also assume that the last column is indexed by v. Consider a realization (H,p) of H obtained from (H', p') by making z and v coincident, that is, letting $p(z) = p'(v)$ and $p(w) = p'(w)$ for all $w \in V'$. Although it is not a proper realization yet, we can consider its projective rigidity matrix and use it to obtain the desired proper realization of H.

The matrix $Q(H,p)$ can be obtained from $Q(H', p')$ by adding the new vertex to the end of the vertex ordering of H', inserting a new column, corresponding to z, next to the column of v and replacing the last j rows by $j+1$ new rows corresponding to the edges in $F \cup \{e\}$. Observe that by the choice of $p(z)$ we can also obtain $Q(H,p)$ from $Q(H', p')$ by inserting the column of z, moving the entries corresponding to F in the column of v to the column of z and then adding a new row corresponding to e. All entries of this new row will be zeros, except the two entries in the columns of v and z. Furthermore, these two entries are x and $-x$, for some non-zero real

number x. Thus by adding the last column of $Q(H,p)$ to its second last column we obtain a block diagonal matrix with $Q(H',p')$ in the upper left block and a non-zero number in the lower right block. Hence $\operatorname{rank} Q(H,p) = \operatorname{rank} Q(H',p') + 1 = |V| - 3$, as required. By perturbing the coordinates slightly, without decreasing the rank, we can then make sure that the vertex coordinates are pairwise different and the realization is proper. $\square$

5.4 Affine rigidity

Gortler et al. [16] introduced the concept of affine rigidity, where affine constraints are imposed on sets of points, see also [74]. A d-dimensional *affine framework* (H,p) is a pair, where H is a hypergraph and p is a map from $V(H)$ to $\mathbb{R}^d$. Roughly speaking, an affine framework (H,p) is *affinely rigid* in $\mathbb{R}^d$ if every other d-dimensional framework (H,q), for which the positions of the vertices in p of each hyperedge $e \in E(H)$ can be mapped to their positions in q by an affine map of $\mathbb{R}^d$, can be obtained by a single affine map of $\mathbb{R}^d$, see also [16]. Gortler et al. [16] define the *strong affinity matrix* of an affine framework (H,p) with $H = (V,E)$, which has size $|E| \times |V|$, and show that the framework is affinely rigid if and only if the rank of this matrix is equal to $|V| - (d+1)$. Thus we may call a hypergraph H *generically affinely rigid* in $\mathbb{R}^d$ if there exists an affinely rigid d-dimensional framework on H, or equivalently, if every generic framework on H is affinely rigid. If, in addition, $|E| = |V| - (d+1)$ then H is said to be *minimally generically affinely rigid* in $\mathbb{R}^d$.

For an integer k and hypergraph H let $B_k(H)$ denote the k-uniform hypergraph whose hyperedges are all those k-element subsets of the vertex set that are contained in some hyperedge of H. It is not hard to see that a generic affine framework (H,p) is affinely rigid in $\mathbb{R}^d$ if and only if the associated framework $(B_{d+2}(H),p)$ is affinely rigid in $\mathbb{R}^d$. Thus it suffices to consider $(d+2)$-uniform hypergraphs.

There is also a strong connection between affine rigidity and a problem from polyhedral scene analysis. One can interpret each hyperedge of a planar affine framework as a planar polygon and say that the framework is sharp if each vertex can be given a third coordinate, such that, in the resulting three dimensional drawing, each polygon remains planar, and the faces do not all lie in a single plane. (See Chapter 7 for formal definitions.) The concept of sharpness can easily be generalized to arbitrary dimension. Whiteley [64] showed that a framework has a non-trivial lifting if and only if it is not affinely rigid. Furthermore, the combinatorial characterization of sharpness given in [64, Theorem 4.2] implies that a $(d+2)$-uniform hypergraph is minimally generically affinely rigid in $\mathbb{R}^d$ if and only if it is $(1,d+1)$-tight.

Thus an immediate corollary of Theorem 5.2.8 is an inductive construction of the 4-uniform minimally generically affinely rigid hypergraphs in the plane.

Theorem 5.4.1 *Let $H = (V, E)$ be a 4-uniform hypergraph. Then H is minimally generically affinely rigid in $\mathbb{R}^2$ if and only if it can be obtained from a single hyperedge of size four by a sequence of 0-extensions, 1-extensions, and 2-extensions.*

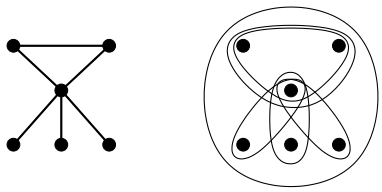


Figure 5.2: A graph G and its neighbourhood hypergraph $N(G)$.

In order to deduce some further combinatorial results we recall that the edge sets of the sparse subhypergraphs of a hypergraph correspond to the independent sets of a matroid. These matroids, which can be defined for all sparsity parameters, are called count matroids. Their rank function is known. See Frank [12] and Whiteley [68] for more details as well as [53] for some related algorithmic problems. Let $H = (V, E)$ be a hypergraph. We shall focus on $(1, k)$-sparsity, which defines the count matroid $\mathcal{M}_{1,k}(H)$ with ground-set E and rank function $r_{1,k}$. A *cover* of $H = (V, E)$ is a collection $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V, each of size at least $k+1$, such that $E = \cup_{i=1}^{t} E_H(X_i)$, where $E_H(X)$ denotes the set of edges of H induced by vertex-set X. We say that a cover is *s-thin* if for each pair of distinct members $X_i, X_j \in \mathcal{X}$ we have $|X_i \cap X_j| \leq s$. If H is $(k+1)$-uniform then the rank of H can be expressed in the following simple form (see [12, Section 13.5]):

$$r_{1,k}(E) = \min \sum_{X \in \mathcal{X}} (|X| - k), \tag{5.3}$$

where the minimum is taken over all $(k-1)$-thin covers $\mathcal{X}$ of H. It follows from the above discussion that a $(d+2)$-uniform hypergraph H is generically affinely rigid in $\mathbb{R}^d$ if and only if $r_{1,d+1}(E) = |V| - (d+1)$.

Zha and Zhang [74] found the following sufficient condition for generic affine rigidity, for which we can give a short proof, using the above rank formula. We say that $H = (V, E)$ is $(d+1)$-*linked* if for each pair $e, e' \in E$ there is a sequence of hyperedges of H, starting with e and ending with e', such that consecutive pairs of hyperedges in the sequence share at least $d+1$ vertices.

Theorem 5.4.2 *[74] Let H be a $(d+1)$-linked hypergraph without isolated vertices. Then H is generically affinely rigid in $\mathbb{R}^d$.*

Proof. It is easy to see that H is $(d+1)$-linked if and only if $B_{d+2}(H)$ is $(d+1)$-linked. Thus we may assume that $H = (V, E)$ is $(d+2)$-uniform. Suppose that H is not generically affinely rigid in $\mathbb{R}^d$. By using (5.3) this implies that there is a d-thin cover $\mathcal{X} = \{X_1, \ldots, X_k\}$ of H with $\sum_{i=1}^{k}(|X_i| - (d+1)) \leq |V| - (d+1) - 1$. Pick an edge e of H and let, say, X_1 be a member of $\mathcal{X}$ that contains e. Since $\mathcal{X}$ is d-thin and H is $(d+1)$-linked we obtain that X_1 contains all edges of H. The fact that H has no isolated vertices gives $X_1 = V$, a contradiction. □

A different sufficient condition for affine rigidity was given by Gortler et al. [16]. Given a graph G, define its *neighbourhood hypergraph*, denoted by $N(G)$, on the same set of vertices as follows: for each vertex v in G add a hyperedge to $N(G)$ consisting of v and its neighbours in G. See Figure 5.2.

Theorem 5.4.3 *[16] Let $G = (V, E)$ be a $(d+1)$-connected graph. Then the neighbourhood hypergraph of G is generically affinely rigid in $\mathbb{R}^d$.*

Next we give a purely combinatorial proof for the two-dimensional version of Theorem 5.4.3 by verifying the following result. The original proof uses, among others, non-symmetric stress matrices and rubber band embeddings. Our proof method was inspired by [35].

Theorem 5.4.4 *Let $G = (V, E)$ be a 3-connected graph. Then*

$$r_{1,3}(B_4(N(G)) = |V| - 3.$$

Proof. Suppose, for a contradiction, that there is a 3-connected graph G for which $r_{1,3}(B_4(N(G)) \leq |V| - 4$. Choose a counterexample $G = (V, E)$ for which $|V|$ is as small as possible and within the family of counterexamples of this size, $|E|$ is as large as possible. Let $H = B_4(N(G))$. The rank formula (5.3) implies that there is a 2-thin cover $\mathcal{X} = \{X_1, \ldots, X_k\}$ of H for which

$$\sum_{i=1}^{k}(|X_i| - 3) \leq |V| - 4 \tag{5.4}$$

holds. We say that a set $X_i \in \mathcal{X}$ is a *core* of some vertex $v \in V$ if $N_G(v) \cup \{v\} \subseteq X_i$.

Claim 5.4.5 *Each vertex $v \in V$ has a unique core.*

Proof. Since $\mathcal{X}$ covers H, each set $\{v, v_1, v_2, v_3\} \in F$ with $\{v_1, v_2, v_3\} \subseteq N_G(v)$ is covered by some $X_j \in \mathcal{X}$. By using the fact that $\mathcal{X}$ is 2-thin, we can deduce that there must be a unique set $X_i \in \mathcal{X}$ that contains v as well as all neighbours of v in G. □

We may also assume that $\mathcal{X}$ is chosen so that the left hand side of (5.4) is minimized. Suppose that $G[X_j]$ is disconnected for some $1 \leq j \leq k$ and consider the family $\mathcal{X}'$ obtained from $\mathcal{X}$ by replacing X_j with the vertex sets of those components of $G[X_j]$ that contain at least four vertices. Observe that $\mathcal{X}'$ is also a (2-thin) cover of H: each hyperedge e in H is a 4-element subset of $N_G(w) \cup \{w\}$ for some vertex w, and hence either the core X_i of w in $\mathcal{X}$ stays in $\mathcal{X}'$ and covers e (if $i \neq j$) or one of the new smaller sets will be a core of w in $\mathcal{X}'$ (if $i = j$). Since $\mathcal{X}'$ would give rise to a strictly smaller value on the left hand side of (5.4), it follows that $G[X_j]$ is connected for all $1 \leq i \leq k$.

For each $v \in V$ let $b(v)$ denote the number of those members of $\mathcal{X}$ that contain v.

Claim 5.4.6 *$b(v) \geq 2$ for every $v \in V$.*

Proof. Suppose, for a contradiction, that some vertex $v \in V$ is in X_1, say, but it is disjoint from X_i for all $2 \leq i \leq k$. It follows from Claim 5.4.5 that for each vertex $v_j \in N_G(v)$ we must have $N_G(v_j) \subseteq X_1$. This implies, by the maximality of $|E|$, that $G[N_G(v)]$ is a complete subgraph of G. It also implies that $|X_1| \geq 5$ unless G is a complete graph on four vertices, for which the theorem is trivially true.

Let $\mathcal{X}' = \{X_1', X_2', \ldots, X_k'\}$, where $X_1' = X_1 - v$ and $X_i' = X_i$ for all $2 \leq i \leq k$. Then $\mathcal{X}'$ is a cover of $B_4(N(G - v))$ satisfying

$$\sum_{i=1}^{k}(|X_i'| - 3) = \sum_{i=1}^{k}(|X_i| - 3) - 1 < |V| - 4 = |V(G - v)| - 3.$$

By the minimality of $|V|$ the graph $G - v$ is not a counterexample to the statement of the theorem, so it follows that $G - v$ is not 3-connected, that is, the graph $G - \{v, x, y\}$ is disconnected for some pair of vertices $x, y \in V$. But G is 3-connected, so v must have at least one neighbour in each connected component of $G - \{v, x, y\}$. This contradicts the fact that $G[N_G(v)]$ is complete. $\square$

Claim 5.4.7 *Suppose that $b(v) \leq 3$ and let $X \in \mathcal{X}$ be the core of v. Then $|X| \geq 6$.*

Proof. First suppose $b(v) = 2$. Let Y be the other member of $\mathcal{X}$ containing v. Since $G[Y]$ is connected, it must contain a neighbour of v in G. The cover is 2-thin, so this implies that $X \cap Y = \{v, y\}$ for some $y \in N_G(v)$. In the subgraph $G[Y]$ vertex v has degree one, hence y must have a neighbour in G which belongs to $Y - X$. It follows that Y is the core of y. Since G is 3-connected, v has at least three neighbours in G. Suppose that $\{a, b\} \subseteq N_G(v) - Y$. Since $b(v) = 2$ the core of a and b must also be

X. The fact that Y is the core of y implies that y cannot be adjacent to a or b. Now the 3-connectivity of G gives that $|X - \{v, y, a, b\}| \geq 2$, from which $|X| \geq 6$ follows.

Next suppose $b(v) = 3$. Let Y, Z be the other members of $\mathcal{X}$ containing v. As above, we obtain that $X \cap Y = \{v, y\}$ and $X \cap Z = \{v, z\}$ for distinct vertices $y, z \in N_G(v)$ and that Y is the core of y and Z is the core of z. Let $a \in (N_G(v) - \{v, y, z\})$. Since $b(v) = 3$, X is the core of a. Using that a cannot be adjacent to y or z we get that a has at least two more neighbours in X and hence $|X| \geq 6$ follows. $\square$

Claim 5.4.8 *Suppose that $b(v) \geq 3$. Then $\sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq 1$.*

Proof. Since $|X_i| \geq 4$ for all $1 \leq i \leq k$, the claim follows immediately if $b(v) \geq 4$. Now suppose that $b(v) = 3$. By Claim 5.4.7 we get $\sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq \left(1 - \frac{3}{4}\right) + \left(1 - \frac{3}{4}\right) + \left(1 - \frac{3}{6}\right) = 1$. $\square$

To obtain a similar bound for the vertices with $b(v) = 2$, at least on average, we have to deal with them together and we need a more careful counting argument. Let $J = \{vx \in E : b(v) = 2,$ and for some pair $X, Y \in \mathcal{X}$ we have $X \cap Y = \{v, x\}\}$, let $W = V(J)$ and $Z_1, Z_2, ..., Z_\ell$ be the vertex sets of the components of the graph $K = (W, J)$. Observe that each vertex with $b(v) = 2$ belongs to W and that each component of K is a star in which each leaf vertex v has $b(v) = 2$. Furthermore, if a component is a star on at least three vertices with center vertex y then the cores of its leaves are pairwise different and the core of y must contain all vertices of this component.

Claim 5.4.9

$$\sum_{v \in W} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq |W|.$$

Proof. It suffices to show that $\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq |Z_j|$ for all $1 \leq j \leq \ell$. Consider the component on Z_j. First suppose $|Z_j| \geq 4$. Then, by using Claim 5.4.7, we can give a lower bound on the contributions of the $|Z_j| - 1$ leaves and the center vertex as follows:

$$\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq (|Z_j| - 1)\left(\frac{1}{2} + 1 - \frac{3}{|Z_j|}\right) +$$

$$+(|Z_j| - 1)\frac{1}{2} + \left(1 - \frac{3}{|Z_j|}\right) \geq |Z_j|,$$

as required.

Now suppose that $|Z_j| = 3$. If $b(c) \geq 4$ for the center vertex c of the star then, by using Claim 5.4.7 again, we obtain $\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq 2\left(\frac{1}{2} + \frac{1}{4}\right) +$

$\left(2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}\right) = 3$, as claimed. If $b(c) = 3$ then $\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq 2 \cdot 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} > 3$ follows.

Finally, suppose that $|Z_j| = 2$. Then either both vertices in Z_j are contained by at most three sets of $\mathcal{X}$, in which case $\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq 2(\frac{1}{2} + \frac{1}{2}) = 2$, or one of them, say c, has $b(c) \geq 4$. In the latter case we get $\sum_{v \in Z_j} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$. This completes the proof of the claim. $\square$

The proof of the theorem follows by using Claims 5.4.8, 5.4.9 and the fact that $b(v) \geq 3$ for all $v \in V - W$:

$$\sum_{i=1}^{k} (|X_i| - 3) = \sum_{i=1}^{k} |X_i| \left(1 - \frac{3}{|X_i|}\right) = \sum_{v \in W} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) +$$

$$+ \sum_{v \in V - W} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq |W| + |V - W| = |V|,$$

contradicting (5.4). $\square$

It may also be possible to use a similar method to deduce the higher dimensional versions of Theorem 5.4.3 but the proof gets more complicated. On the other hand, there is an even simpler combinatorial proof in the case when $d = 1$. It is based on the fact that every 2-connected graph has an ear-decomposition and uses induction on the number of ears. Each new ear added to G generates a set of new hyperedges in $B_3(N(G))$. By using Lemma 5.2.1 one can obtain a sufficiently large $(1, 2)$-tight spanning subhypergraph of the extended hypergraph. We omit the details.

Chapter 6

Gain-sparsity and Symmetry-forced Rigidity in the Plane

This chapter deals with finite bar-and-joint frameworks with point group symmetry in the *symmetry-forced* setting and extends Laman's classical theorem as well as its matroidal background and algorithmic implications, to planar frameworks with rotational or dihedral symmetry, assuming that the joint positions are as generic as possible subject to the symmetry conditions. In our symmetry-forced setting, a framework is said to be *symmetry-forced flexible* if it has a non-trivial *symmetric* infinitesimal motion. For the symmetry-generic frameworks that we consider, this is equivalent to the existence of a non-trivial symmetry preserving flex [48], and our main result characterizes symmetric frameworks that admit nontrivial symmetry preserving flexes in terms of simple count conditions of the underlying quotient group-labeled graphs, which can be checked in polynomial time by combinatorial algorithms.

By using the orbit rigidity matrix introduced by Schulze and Whiteley [51], we can reformulate our problems in terms of the generic rank of a matrix in which each row corresponds to an edge orbit and each vertex orbit has two columns. This in turn is equivalent to characterizing independence in a matroid defined on the edge set of the group-labeled quotient graph, in which vertices and edges correspond to vertex and edge orbits, respectively, and which concisely represents the graph structure with the corresponding symmetry. Our main results characterize these matroids in the case of rotation symmetry or dihedral symmetry $\mathcal{D}_k$ of order $2k$ with odd k. If the underlying symmetry is cyclic, the matroid turns out to be a (k,l)-*gain-count matroid*, in which independence is defined by imposing certain sparsity conditions on the edge sets of a graph, whose edges are labeled by group elements. In the dihedral case the matroid arises by a related, but more general construction.

We prove our results by developing Henneberg type inductive constructions for the bases of our matroids and show that these operations preserve the row-independence of the orbit rigidity matrix. In our problems, due to the more complex sparsity conditions and the group labeling, we also need some new operations and extended geometric arguments, to handle the symmetry constraints.

The complete answer in the case of dihedral symmetry remains open. However, most of our inductive steps (extending or reducing a symmetric framework or a labeled graph, respectively) are valid also for dihedral groups $\mathcal{D}_k$ with even k, and can be used to show that in the even case the irreducible graphs (frameworks), where our reduction operations are not applicable, are very special. Interestingly, the smallest such framework, which is predicted to be rigid by the matroidal count but is flexible is the Bottema mechanism, a well-known mechanism in the kinematics literature (see, e.g., [70]).

For the case when the underlying symmetry is cyclic, the same combinatorial characterizations were also given by Malestein and Theran [38], [39] by a completely different proof approach. The main contributions of this paper are (i) to develop a concise approach to analyze the rigidity of symmetric frameworks based on inductive constructions and (ii) to give the first combinatorial characterization for non-cyclic symmetry in the plane, which is far complicated than cyclic case. After publishing the technical report [27] of this paper, our formulation and results on inductive constructions were used for analyzing the infinitesimal rigidity of symmetric frameworks [50] or the symmetric-forced rigidity of symmetric frameworks on surfaces [42].

Now let us introduce notations used throughout this chapter.

Let E be a finite set. A *partition* $\mathcal{P}$ of E is a family of nonempty subsets of E such that each element of E belongs to exactly one member of $\mathcal{P}$. If $E = \emptyset$, the partition of E is defined as the empty set. A *subpartition* of E is a partition of a subset of E.

A vertex subset $X \subset V(G)$ (resp., an edge subset $X \subset E(G)$ is called a *separator* (resp., a *cut*) if the removal of X disconnects G. A separator (resp., a cut) is called *nontrivial* if its removal disconnects G into at least two nontrivial connected components, where a connected component is called trivial if it consists of a single vertex. G is called *essentially k-connected* (resp., *essentially k-edge-connected*) if the size of any nontrivial separator (resp., any nontrivial cut) is at least k.

For simplicity, some properties of edge-induced subgraphs will be associated with the corresponding edge sets as follows. Let $F \subseteq E$. F is called *connected* if $G[F]$ is connected. A *connected component* of F is the edge set of a connected component

of $G[F]$. $C(F)$ denotes the partition of F into connected components of F, and let $c(F) = |C(F)|$.

Let $G = (V, E)$ be a directed graph. A *walk* in G is a sequence $W = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ of vertices and edges such that v_{i-1} and v_i are the endvertices of e_i for every $1 \leq i \leq k$. We often denote a walk as a sequence of edges implicitly assuming the incidence at each vertex. For two walks W and W' for which the end vertex of W and the starting vertex of W' coincide, we denote the concatenation of W and W' (that is, the walk W followed by W') by $W * W'$. A walk is called *closed* if the starting vertex and the end vertex coincide.

It is sometimes convenient to regard the empty set as a subgroup of a group. Let $\mathcal{D}$ be a dihedral group. For a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$, $\bar{\mathcal{C}}$ denotes the maximal cyclic subgroup containing $\mathcal{C}$.

We refer the reader to [27] for the proofs not presented in this work.

6.1 Gain Graphs

In this section we shall review some basic properties of gain graphs. We refer the reader to [19, 72, 73] for more details.

Let $G = (V, E)$ be a directed graph which may contain multiple edges and loops, and let $\mathcal{S}$ be a group. An *$\mathcal{S}$-gain graph* (G, ϕ) is a pair, in which each edge is associated with an element of $\mathcal{S}$ by a *gain function* $\phi : E \to \mathcal{S}$. The orientation of G is, in some sense, arbitrary, and is used only as a reference orentation: the orientation of each edge may be changed, provided that we also modify ϕ such that if the edge has gain g in one direction then it has gain g^{-1} in the other direction. Therefore we often do not distinguish between G and the underlying undirected graph.

Let W be a walk in (G, ϕ). The *gain* of W is defined as $\phi(W) = \phi(e_1) \cdot \phi(e_2) \cdots \phi(e_k)$ if each edge is oriented in the forward direction through W, and for a backward edge e_i we replace $\phi(e_i)$ with $\phi(e_i)^{-1}$ in the product. Note that $\phi(W^{-1}) = \phi(W)^{-1}$.

Let (G, ϕ) be a gain graph. For $v \in V(G)$ we denote by $\pi_1(G, v)$ the set of closed walks starting at v. Similarly, for $X \subseteq E(G)$ and $v \in V(G)$, $\pi_1(X, v)$ denotes the set of closed walks starting at v and using only edges of X, where $\pi_1(X, v) = \emptyset$ if $v \notin V(X)$.

Let $X \subseteq E(G)$. The subgroup induced by X relative to v is defined as $\langle X \rangle_{\phi,v} = \{\phi(W) \mid W \in \pi_1(X, v)\}$. The subscript ϕ of $\langle X \rangle_{\phi,v}$ is sometimes omitted if it is clear from the context. Note that, for any connected $X \subseteq E(G)$ and two vertices $u, v \in V(X)$, $\langle X \rangle_u$ is conjugate to $\langle X \rangle_{\psi,v}$. (See, e.g., [19, page 88] for the proof.)

6.1.1 The switching operation

For $v \in V(G)$ and $g \in \mathcal{S}$, a *switching operation at v with g* changes the gain function ϕ on $E(G)$ as follows.

$$\phi'(e) = \begin{cases} g \cdot \phi(e) \cdot g^{-1} & \text{if } e \text{ is a loop incident with } v \\ g \cdot \phi(e) & \text{if } e \text{ is a non-loop edge and is directed from } v \\ \phi(e) \cdot g^{-1} & \text{if } e \text{ is a non-loop edge and is directed to } v \\ \phi(e) & \text{otherwise.} \end{cases} \tag{6.1}$$

We say that a gain function ϕ' on edge set $E(G)$ is *equivalent* to another gain function ϕ on $E(G)$ if ϕ' can be obtained from ϕ by a sequence of switching operations.

The following two facts are fundamental. (See, e.g., [19, Section 2.5.2] or [72, Section 5] for the proofs.)

Proposition 6.1.1 *Let (G, ϕ) be a gain graph. Let ϕ' be the gain function obtained from ϕ by a switching operation. Then, for any $X \subseteq E(G)$ and $u \in V(G)$, $\langle X \rangle_{\phi',u}$ is conjugate to $\langle X \rangle_{\phi,u}$.*

Proposition 6.1.2 *Let (G, ϕ) be a gain graph. Then, for any forest $F \subseteq E(G)$, there is a gain function ϕ' equivalent to ϕ such that $\phi'(e) = \mathrm{id}$ for every $e \in F$.*

6.1.2 Balanced and cyclic sets of edges

As we shall see, the subgroup $\langle X \rangle_{\psi,v}$ itself will not be important, when we define our matroids induced by gains. We only need to know whether $\langle X \rangle_{\psi,v}$ is trivial or not, or whether it is cyclic or not. We now introduce notions to describe these properties.

Let (G, ϕ) be a gain graph. An edge subset $F \subseteq E(G)$ is called *balanced* if $\langle F \rangle_{\psi,v}$ is trivial for every $v \in V(F)$. Note that F is balanced if and only if every cycle in F is balanced. The latter property is the definition of the balancedness given by Zaslavsky [72].

In the same way, an edge subset $F \subseteq E(G)$ is called *cyclic* if $\langle F \rangle_{\psi,v}$ is cyclic for every $v \in V(F)$. (Note that the terms balanced and cyclic are not exclusive.) A gain graph (G, ϕ) is called *balanced* and *cyclic* if $E(G)$ is balanced and cyclic, respectively.

Proposition 6.1.2 suggests a simple way to check the above introduced properties of X, in analogy with the fact that the cycle space of a graph is spanned by fundamental cycles. For a connected $X \subseteq E(G)$, take a spanning tree T of the edge induced graph $G[X]$. By Proposition 6.1.2 we can convert the gain function to an equivalent gain function such that $\phi(e) = \mathrm{id}$ for all $e \in T$. Now consider any

closed walk $W \in \pi_1(X, v)$, and denote W by $W = v_1v_2, v_2v_3, \ldots, v_kv_{k+1}$, and let $W_i = P_i * \{v_iv_{i+1}\} * P_{i+1}^{-1}$ for $1 \leq i < k$, where P_i denotes the path from v to v_i in T. Then observe $\phi(W) = \phi(W_1) \cdot \phi(W_2) \cdots \phi(W_k)$. By $\phi(e) = \mathrm{id}$ for all $e \in T$, we deduce that $\phi(W)$ is a product of elements in $\{\phi(e) : e \in X \setminus T\}$, implying that $\langle X \rangle_{\phi,v} \subseteq \langle \phi(e) : e \in X \setminus T \rangle$, where $\langle \phi(e) : e \in X \setminus T \rangle$ is the group generated by $\{\phi(e) : e \in X \setminus T\}$. Conversely, $\phi(e)$ is contained in $\langle X \rangle_{\phi,v}$ for all $e \in X \setminus T$. Thus, $\langle X \rangle_{\phi,v} = \langle \phi(e) : e \in X \setminus T \rangle$. In particular, we proved the following.

Lemma 6.1.3 *For a connected $X \subseteq E(G)$ and a spanning tree T of $G[X]$, suppose that $\phi(e) = \mathrm{id}$ for all $e \in T$. Then, $\langle X \rangle_{\phi,v} = \langle \phi(e) \colon e \in X \setminus T \rangle$. In particular, the following hold.*

(i) *X is unbalanced if and only if there is an edge in $X \setminus T$ whose gain is non-identity.*

(ii) *X is cyclic if and only if all gains of $X \setminus T$ are contained in a cyclic subgroup of $\mathcal{S}$.*

The following technical lemmas will be used in the proof of our main theorem.

Lemma 6.1.4 *Let (G, ϕ) be a $\mathcal{S}$-gain graph, and X and Y be connected edge subsets such that the graph $(V(X) \cap V(Y), X \cap Y)$ is connected.*

(1) *If X and Y are balanced, then $X \cup Y$ is balanced.*

(2) *If X is balanced and Y is cyclic, then $X \cup Y$ is cyclic.*

(3) *If X, Y are cyclic and $X \cap Y$ is unbalanced, then $X \cup Y$ is cyclic, provided that for every non-trivial cyclic subgroup $\mathcal{C}$ of $\mathcal{S}$ there is a unique maximal cyclic subgroup $\bar{\mathcal{C}}$ of $\mathcal{S}$ containing $\mathcal{C}$.*

Proof. Since the graph $(V(X) \cap V(Y), X \cap Y)$ is connected, there is a spanning tree T in $G[X \cup Y]$ such that $T \cap X$ is a spanning tree of $G[X]$, $T \cap Y$ is a spanning tree of $G[Y]$, and $T \cap X \cap Y$ is a spanning tree of $G[X \cap Y]$. By Proposition 6.1.2, there is a gain function ϕ' equivalent to ϕ such that $\phi'(e) = \mathsf{id}$ for each $e \in T$.

If X and Y are balanced, Lemma 6.1.3 implies that $\phi'(e) = \mathsf{id}$ for all $e \in X \cup Y$. Thus (1) holds.

If X is balanced, then every label in $X \cup Y$ is contained in $\langle Y \rangle_{\phi',v}$ by Lemma 6.1.3, and hence $X \cup Y$ is cyclic if Y is cyclic. This implies (2).

If X, Y are cyclic and $X \cap Y$ is unbalanced, then there is an edge $e \in X \cap Y$ for which $\phi'(e)$ is non-identity. Let $\mathcal{C}$ be a cyclic subgroup of $\mathcal{S}$ generated by $\phi'(e)$ and $\bar{\mathcal{C}}$ be the maximal cyclic subgroup containing $\mathcal{C}$. Since X and Y are cyclic, Lemma 6.1.3 implies that $\phi'(e) \in \bar{\mathcal{C}}$ holds for every $e \in X$ and for every $e \in Y$. Therefore $X \cup Y$ is cyclic. $\square$

Lemma 6.1.5 *Let (G, ϕ) be a gain graph, and X and Y be connected balanced edge subsets. If the number of connected components of the graph $(V(X) \cap V(Y), X \cap Y)$ is two, then $X \cup Y$ is cyclic.*

Proof. We take a spanning tree T of $G[X \cup Y]$ such that $T \cap X$ is a spanning tree of $G[X]$. Since the number of connected components of $(V(X) \cap V(Y), X \cap Y)$ is two, $T \cap Y$ consists of two connected components, denoted T_1 and T_2. $\{V(T_1), V(T_2)\}$ partitions Y into three subsets $\{Y_1, Y_2, Y_3\}$ such that $Y_i = \{e \in Y : V(\{e\}) \subseteq V(T_i)\}$ for $i = 1, 2$ and $Y_3 = Y \setminus (Y_1 \cup Y_2)$.

By Proposition 6.1.2, we can take a gain function ϕ' equivalent to ϕ such that $\phi'(e) = \mathsf{id}$ for $e \in T$. Since X and Y are balanced, we have $\phi'(e) = \mathsf{id}$ for $e \in X \cup Y_1 \cup Y_2$. Moreover, assuming that every edge in Y_3 is oriented toward $V(Y_1)$, we have $\phi'(e) = \phi'(f)$ for all $e, f \in Y_3$, since otherwise $T_1 \cup T_2 \cup \{e, f\}$ contains an unbalanced cycle, contradicting the fact that Y is balanced. Therefore $X \cup Y$ is cyclic. $\square$

6.2 Gain Count Matroids

6.2.1 Matroids induced by submodular functions

Let E be a finite set. A function $\mu : 2^E \to \mathbb{R}$ is called *submodular* if $\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y)$ for every $X, Y \subseteq E$. μ is *monotone* if $\mu(X) \leq \mu(Y)$ for any $X \subseteq Y$. A monotone submodular function $\mu : 2^E \to \mathbb{Z}$ *induces* a matroid on E, where $F \subseteq E$ is independent if and only if $|I| \leq \mu(I)$ for every nonempty $I \subseteq F$. See e.g. [12, Section 13.4]. This matroid is denoted by $\mathcal{M}(\mu)$.

For a monotone submodular function μ, let $\nu = \mu - 1$. Then, ν is monotone submodular and induces the matroid $\mathcal{M}(\nu)$. This matroid is referred to as the *Dilworth truncation* of $\mathcal{M}(\mu)$. Although the details are omitted here, the name of Dilworth truncation is justified from a connection with Dilworth truncation for general matroids, see [12] for more details.

Now we consider the union of two matroids induced by monotone submodular functions μ_1 and μ_2. Since monotonicity and submodularity are both preserved under the sum operation, $\mu_1 + \mu_2$ is monotone and submodular. In general, the union of $\mathcal{M}(\mu_1)$ and $\mathcal{M}(\mu_2)$ is not equal to $\mathcal{M}(\mu_1 + \mu_2)$. We do have equality in some special cases, for example, when $\mu_1 = \mu_2$ or when both μ_1 and μ_2 are nonnegative.

As an example, consider the union of two copies of the graphic matroid of a graph $G = (V, E)$. It is the matroid induced by $f_{2,2}$ defined by $f_{2,2}(F) = 2|V(F)| - 2$ on 2^E, as $f_{2,2}/2$ induces the graphic matroid on G. The 2-dimensional generic rigidity

matroid is the one induced by $f_{2,2} - 1$, and hence it is the Dilworth truncation of the union of two copies of the graphic matroid.

In general, for a graph $G = (V, E)$ and two integers k and l with $k \geq 1$ and $l \leq 2k - 1$, let

$$f_{k,l}(F) = k|V(F)| - (l - k) \qquad (F \subseteq E).$$

G is called (k, l)-*sparse* if $|F| \leq f_{k,l}(F)$ for any nonempty $F \subseteq E$. The matroid induced by $f_{k,l}$ is the (k, l)-count matroid on G. If $l \geq 0$, $\mathcal{M}(f_{k,l})$ is indeed the one induced by $f_{k,0}$, truncated l times. See e.g. [12] for more detail. Below we shall apply the same construction to the union of some copies of a frame matroid to define gain-count matroids.

6.2.2 Gain-count matroids

We shall consider frame matroids on gain graphs. Let $\mathcal{S}$ be a group and (G, ϕ) be an $\mathcal{S}$-gain graph. The *frame matroid* of (G, ϕ) is defined such that $F \subseteq E$ is independent if and only if each connected component of F contains no cycle or just one cycle, which is unbalanced if exists [73]. If we define $g_{\mathcal{S}} : 2^E \to \mathbb{Z}$ by

$$g_{\mathcal{S}}(F) = \sum_{F_i \in C(F)} (|V(F_i)| - 1 + \alpha_{\mathcal{S}}(F_i)) \tag{6.2}$$

where

$$\alpha_{\mathcal{S}}(F) = \begin{cases} 1 & \text{if } F \text{ is unbalanced} \\ 0 & \text{otherwise,} \end{cases} \tag{6.3}$$

then the frame matroid is the matroid induced by $g_{\mathcal{S}}$. We omit the subscript $\mathcal{S}$ from $\alpha_{\mathcal{S}}$ if it is clear from the context.

For an $\mathcal{S}$-gain graph and two positive integers k and l with $k \leq l$, we define $g_{k,l} : 2^E \to \mathbb{Z}$ by

$$g_{k,l}(F) = kg_{\mathcal{S}}(F) - (l - k) \qquad (F \subseteq E). \tag{6.4}$$

We call the matroid $\mathcal{M}(g_{k,l})$ induced by $g_{k,l}$ a (k, l)-*gain-count matroid* or *g-count matroid* for short. This matroid is the union of k copies of the frame matroid, followed by $l - k$ Dilworth truncations. In this paper, we shall investigate the $(2, 3)$-g-count matroid and its variants.

The independence of $\mathcal{M}(g_{k,l})$ can be described in a compact form. (See [27] for the proof, which is a rather straightforward calculation.)

Lemma 6.2.1 *Let (G, ϕ) be an $\mathcal{S}$-gain graph with $G = (V, E)$. Then E is independent in $\mathcal{M}(g_{k,l})$ if and only if $|F| \leq k|V(F)| - l + k\alpha(F)$ for any nonempty $F \subseteq E$.*

In this sense, we may define (k,l)-*gain-sparsity* as in the case of (k,l)-sparsity of undirected graphs as follows.

Definition 6.2.2 *Let k and l be positive integers with $k \leq l$ and (G,ϕ) be an $\mathcal{S}$-gain graph with a graph $G = (V,E)$ and a group $\mathcal{S}$. An edge set $X \subseteq E$ is called* (k,l)-gain-sparse *(or* (k,l)-g-sparse *for short) if $|F| \leq g_{k,l}(F)$ for any nonempty $F \subseteq X$, i.e.,*

- $|F| \leq k|V(F)| - l$ *for every nonempty balanced $F \subseteq X$;*
- $|F| \leq k|V(F)| - l + k$ *for every nonempty unbalanced $F \subseteq X$,*

and it is clalled (k,l)-gain-tight *(or* (k,l)-g-tight *for short) if it is (k,l)-g-sparse with $|X| = g_{k,l}(X)$.*

(G,ϕ) is called (k,l)-g-sparse *if so is E, and it is called* maximum (k,l)-g-tight *if it is (k,l)-g-sparse with $|E| = k|V| - l + k$.*

Remark 6.2.3 *Note that the value of $g_{k,l}$ is invariant under switching operations, and thus the induced matroid is uniquely determined up to equivalence of gain functions.*

6.3 Constructive characterization of maximum (2.3)-g-tight graphs

6.3.1 Operations preserving (2,3)-g-sparsity

In this section we define three operations, called *extensions*, that preserve $(2,3)$-g-sparsity. The first two operations generalize the Henneberg operations [60, 68] to gain graphs.

Let (G,ϕ) be an $\mathcal{S}$-gain graph. The *0-extension* adds a new vertex v and two new non-loop edges e_1 and e_2 to G such that the new edges are incident to v and the other endvertices are two not necessarily distinct vertices of $V(G)$. If e_1 and e_2 are not parallel then their labels can be arbitrary. Otherwise the labels are assigned such that $\phi(e_1) \neq \phi(e_2)$, assuming that e_1 and e_2 are directed to v.

The *1-extension* first chooses an edge e and a vertex z, where e may be a loop and z may be an endvertex of e. It subdivides e, with a new vertex v and new edges e_1, e_2 such that the tail of e_1 is the tail of e and the tail of e_2 is the head of e. The labels of the new edges are assigned such that $\phi(e_1)\cdot\phi(e_2)^{-1} = \phi(e)$. The 1-extension also adds a third edge e_3 oriented to v. The label of e_3 is assigned so that it is *locally unbalanced*, i.e., every two-cycle e_ie_j, if exists, is unbalanced.

The *loop 1-extension* adds a new vertex v to G and connects it to a vertex $z \in V(G)$ by a new edge with any label. It also adds a new loop l incident to v with $\phi(l) \neq \mathrm{id}$.

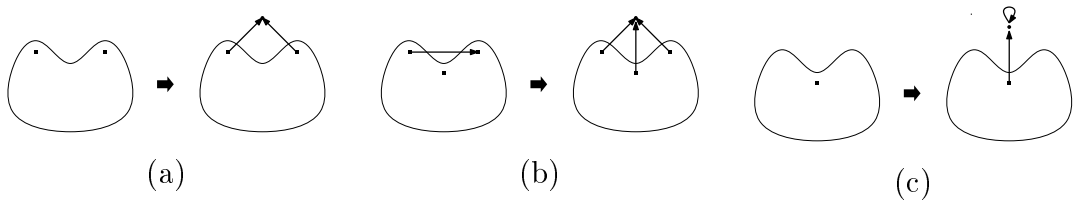


Figure 6.1: (a) 0-extension, where the new edges may be parallel. (b) 1-extension, where the removed edge may be a loop and the new edges may be parallel. (c) loop-1-extension.

The 0-extension and the 1-extension were already considered by Ross [45] for $\mathbb{Z}^2$-gain graphs. In the covering graph each operation can be seen as a graph operation that preserves the underlying symmetry. Some of them can be recognized as performing Henneberg operations [60, 68] simultaneously. In case of 3-fold rotation symmetry, these operations are considered by Schulze [48].

Lemma 6.3.1 *Let (G, ϕ) be a (2,3)-g-sparse $\mathcal{S}$-gain graph. Applying the 0-extension, 1-extension or loop 1-extension to G results in a (2,3)-g-sparse graph (G', ϕ') with $|V(G')| = |V(G)| + 1$ and $|E(G')| = |E(G)| + 2$.*

Proof. For a contradiction, suppose that G' contains an edge set $F \subseteq E(G')$ for which $|F| > 2|V(F)| - 3 + 2\alpha(F)$. Let v be the new vertex added by the extension, and let E_v be the set of edges incident to v. Since $E(G') \setminus E_v \subseteq E(G)$, $E_v \cap F \neq \emptyset$. In particular, $v \in V(F)$. Also, since the new labeling is assigned to be locally unbalanced, F is not contained in E_v.

If G' is constructed by a 1-extension then let e be the subdivided edge of G and let e_1 and e_2 be the resulting two new edges.

Let $F' = F \setminus E_v$. If G' is constructed by a 1-extension and $\{e_1, e_2\} \subseteq F$, then we further insert e to F'. We then have $|F'| \geq |F| - 2$, $|V(F')| = |V(F)| - 1$, and $\alpha(F') \leq \alpha(F)$ in each case. These imply $|F'| \geq |F| - 2 > 2|V(F)| - 5 + 2\alpha(F) \geq 2|V(F')| - 3 + 2\alpha(F')$, contradicting the $(2,3)$-g-sparsity of G as $\emptyset \neq F' \subseteq E(G)$. $\square$

We shall define the inverse moves of the operations above, which are called *reductions*. For a vertex v and two incoming non-loop edges $e_1 = (u, v)$ and $e_2 = (w, v)$, we denote by $e_1 \cdot e_2^{-1}$ a new edge from u to w with label $\phi(e_1) \cdot \phi(e_2)^{-1}$ (by extending ϕ). If $u = w$ then $e_1 \cdot e_2^{-1}$ is a loop. Each reduction corresponds to one of the following operations on a gain graph (G, ϕ).

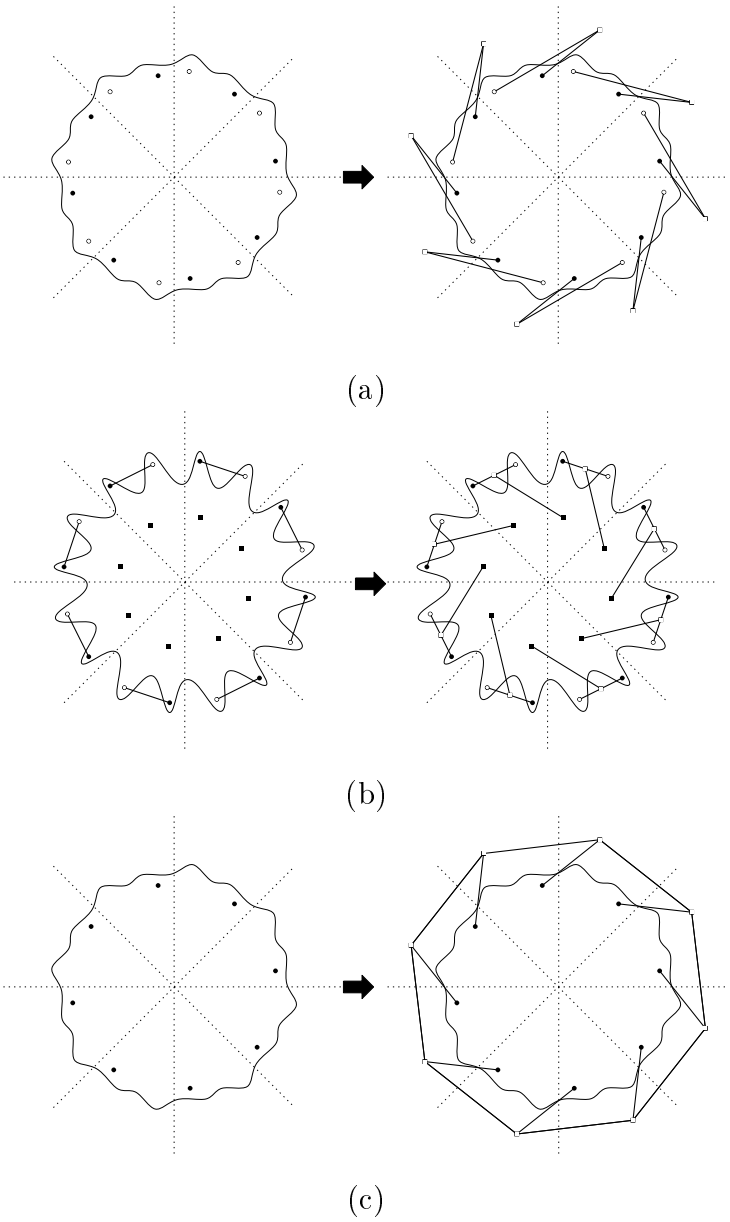


Figure 6.2: (a) 0-extension, (b) 1-extension, (c) loop-1-extension in the covering graph.

The *0-reduction* chooses a degree two vertex and deletes it from G.

The *1-reduction* chooses a vertex v with $d(v) = 3$ that is not incident to a loop. Let e_1, e_2, e_3 be the edges incident to v. Without loss of generality we may assume that each e_i is oriented to v. The 1-reduction deletes v with the incident edges and adds one of $e_1 \cdot e_2^{-1}$, $e_2 \cdot e_3^{-1}$ and $e_3 \cdot e_1^{-1}$ as a new edge.

The *loop 1-reduction* chooses a vertex incident to exactly one loop and one non-loop edge and deletes the chosen vertex with the incident edges.

A 1-reduction may destroy the $(2,3)$-g-sparsity of a graph. We say that a reduction (at a vertex v) is *admissible* if the resulting graph is $(2,3)$-g-sparse.

6.3.2 Constructive characterization

Lemma 6.3.2 *Let (G, ϕ) be a (2,3)-g-sparse graph and $v \in V(G)$ a vertex not incident to a loop with $d(v) = 3$. Then there is an admissible 1-reduction at v.*

Proof. Let $E = E(G)$, $G' = G - v$ and $E' = E(G')$. Let e_1, e_2, e_3 be the edges incident to v in G. Without loss of generality we may assume that each e_i is oriented to v. For simplicity we put $e_{i,j} = e_i \cdot e_j^{-1}$.

Suppose for a contradiction that there is no admissible splitting at v, that is, none of $E' + e_{1,2}$, $E' + e_{2,3}$ and $E' + e_{3,1}$ is independent in $\mathcal{M}(g_{2,3})$. Equivalently, $e_{1,2}, e_{2,3}, e_{3,1} \in \mathrm{cl}_g(E')$, where cl_g denotes the closure operator of $\mathcal{M}(g_{2,3})$. Let $X = \{e_1, e_2, e_3, e_{1,2}, e_{2,3}, e_{3,1}\}$.

Claim 6.3.3 $e_1 \in \mathrm{cl}_g(X - e_1)$.

Proof. We split the proof into three cases depending on the cardinality of $N(v)$.

If $|N(v)| = 3$ then, by Proposition 6.1.2, we may assume $\phi(e_1) = \phi(e_2) = \phi(e_3) = \mathrm{id}$. We then have $\phi(e_{1,2}) = \phi(e_{2,3}) = \phi(e_{3,1}) = \mathrm{id}$. Therefore X forms a balanced K_4, which is a circuit of $\mathcal{M}(g_{2,3})$. Thus, $e_1 \in \mathrm{cl}_g(X - e_1)$ holds.

If $|N(v)| = 2$ then we may assume that e_1 and e_2 are parallel. By Proposition 6.1.2, we may assume that $\phi(e_2) = \phi(e_3) = \mathsf{id}$. This implies $\phi(e_{1,3}) = \phi(e_1)$ and $\phi(e_{2,3}) = \mathsf{id}$. Since G is $(2,3)$-g-sparse, we have $\phi(e_1) \neq \mathsf{id}$ by $\phi(e_2) = \phi(e_3) = \mathsf{id}$, which implies that $e_{1,2}$ is an unbalanced loop with $\phi(e_{1,2}) = \phi(e_1)$. It can be easily checked, by counting, that X is indeed a circuit in $\mathcal{M}(g_{2,3})$. Thus, $e_1 \in \mathrm{cl}_g(X - e_1)$ holds.

If $|N(v)| = 1$ then let $X' = \{e_1, e_2, e_3, e_{1,2}\}$. We have $|X'| = 2|V(X')|$ and X' is a circuit of $\mathcal{M}(g_{2,3})$. Therefore $e_1 \in \mathrm{cl}_g(X' - e_1) \subset \mathrm{cl}_g(X - e_1)$. □

Since $e_{1,2}, e_{2,3}, e_{3,1} \in \mathrm{cl}_g(E')$, by Claim 6.3.3, we have $e_1 \in \mathrm{cl}_g(X - e_1) \subseteq \mathrm{cl}_g(E' + X - e_1) = \mathrm{cl}_g(E' + e_2 + e_3) = \mathrm{cl}_g(E - e_1)$, which contradicts the $(2,3)$-g-sparsity of G. □

The following constructive characterization of maximum $(2,3)$-g-tight graphs is a direct consequence of Lemma 6.3.1 and Lemma 6.3.2. (See [27] for the concrete proof.)

Theorem 6.3.4 *An $\mathcal{S}$-gain graph (G, ϕ) is maximum (2,3)-g-tight if and only if it can be built up from an $\mathcal{S}$-gain graph with one vertex and an unbalanced loop incident to it with a sequence of 0-extensions, 1-extensions, and loop-1-extensions.*

Proof. By Lemma 6.3.1, by applying any of the extension operations we obtain a maximum (2,3)-g-tight graph from a maximum $(2,3)$-g-tight graph.

To prove the other direction it is sufficient to show that G can be reduced to a smaller $(2,3)$-g-tight graph. Since $|E(G)| = 2|V(G)| - 1$, the average degree is less than 4, which implies that there is a vertex v of degree at most 3 and the minimum degree in G is not smaller than two. If $d(v) = 2$, the 0-reduction can be applied at v which is always admissible. If $d(v) = 3$, we have two cases depending on whether v is incident to a loop or not. If v is incident to a loop, the loop-1-reduction, which is always admissible, can be applied at v to obtain a smaller $(2,3)$-g-tight graph. Otherwise, by Lemma 6.3.2, there is an admissible 1-reduction at v. $\square$

6.4 Symmetry-forced rigidity

In this section we define the notion of symmetry-forced infinitesimal rigidity, introduced by Schulze and Whiteley [51]. We have already introduced $\mathcal{S}$-symmetric graphs in Section 1.4.1. In Section 6.4.2 we introduce symmetry-forced infinitesimal rigidity, which is only concerned with infinitesimal motions invariant under the underlying symmetry. In Section 6.4.3 we introduce the orbit rigidity matrix, which is the main tool for investigating symmetry-forced infinitesimal rigidity in the subsequent sections. In Section 6.4.4 we prove a necessary condition for symmetric frameworks to be symmetry-forced infinitesimally rigid.

6.4.1 Quotient graphs of $\mathcal{S}$-symmetric graphs

Let H be an $(\mathcal{S}, \rho)$-symmetric graph. The *quotient graph* $H/\mathcal{S}$ of H is a multigraph on the set $V(H)/\mathcal{S}$ of vertex orbits, together with the set $E(H)/\mathcal{S}$ of edge orbits as the edge set. An edge orbit may be represented by a loop in $H/\mathcal{S}$. Figure 6.3 provides an example when $\mathcal{S}$ is a dihedral group.

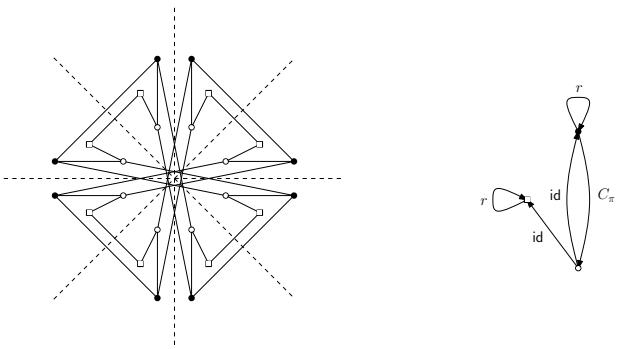


Figure 6.3: A $\mathcal{D}_4$-symmetric graph and the quotient gain graph.

Different graphs may have the same quotient graph. However, if we assume that ρ is free, then a gain labeling makes the relation one-to-one. To see this, we arbitrarily

choose a vertex v as a representative vertex from each vertex orbit. Then, each orbit is written by $\mathcal{S}v = \{gv \colon g \in \mathcal{S}\}$. If ρ is a free action, an edge orbit connecting $\mathcal{S}u$ and $\mathcal{S}v$ in $H/\mathcal{S}$ can be written by $\{\{gu, ghv\} \colon g \in \mathcal{S}\}$ for a unique $h \in \mathcal{S}$. We then orient the edge orbit from $\mathcal{S}u$ to $\mathcal{S}v$ in $H/\mathcal{S}$ and assign to it the gain h. In this way, we obtain *the quotient $\mathcal{S}$-gain graph*, denoted $(H/\mathcal{S}, \phi)$.

Conversely, any $\mathcal{S}$-gain graph (G, ϕ) can be "lifted" as an $(\mathcal{S}, \rho)$-symmetric graph with a free action ρ. To see this, we simply denote the pair (g, v) of $g \in \mathcal{S}$ and $v \in V(G)$ by gv. The *covering graph* (also known as the derived graph) of (G, ϕ) is the simple graph with vertex set $\mathcal{S} \times V(G) = \{gv \colon g \in \mathcal{S}, v \in V(G)\}$ and the edge set $\{\{gu, g\phi(e)v\} \colon e = (u, v) \in E(G), g \in \mathcal{S}\}$. Clearly, $\mathcal{S}$ freely acts on the covering graph, under which the quotient gain graph comes back to (G, ϕ). For more properties of covering graphs, see e.g. [19].

6.4.2 Symmetry-forced infinitesimal rigidity

We shall consider "symmetry-preserving" infinitesimal motions of symmetric frameworks. We say that an infinitesimal motion $m : V(H) \to \mathbb{R}^d$ is *symmetric* if

$$gm(v) = m(gv) \qquad \text{for all } g \in \mathcal{S} \text{ and for all } v \in V(H). \tag{6.5}$$

The set of infinitesimal isometries and the set of infinitesimal motions form a linear space, denoted by $\mathrm{iso}(P)$ and $L(H, p)$, respectively.

The set of $\mathcal{S}$-symmetric infinitesimal motions and the set of trivial ones form linear subspaces of $L(H, p)$ and $\mathrm{tri}(H, p)$, denoted $L_{\mathcal{S}}(H, p)$ and $\mathrm{tri}_{\mathcal{S}}(H, p)$, respectively. We say that (H, p) is *symmetry-forced infinitesimally rigid* if $L_{\mathcal{S}}(H, p) = \mathrm{tri}_{\mathcal{S}}(H, p)$.

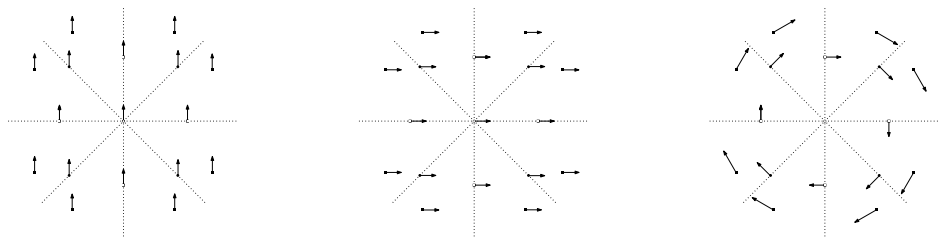
A set P of points is called *$\mathcal{S}$-symmetric* if $gP = \{gp \colon p \in P\} = P$ for all $g \in \mathcal{S}$. An infinitesimal isometry $m : P \to \mathbb{R}^d$ of an $\mathcal{S}$-symmetric point set P is called *$\mathcal{S}$-symmetric* if $gm(x) = m(gx)$ for all $x \in P$ and $g \in \mathcal{S}$. The set of $\mathcal{S}$-symmetric infinitesimal isometries forms a linear subspace of $\mathrm{iso}(P)$, denoted $\mathrm{iso}_{\mathcal{S}}(P)$. Clearly, $\mathrm{tri}_{\mathcal{S}}(H, p)$ is isomorphic to $\mathrm{iso}_{\mathcal{S}}(p(H))$.

Example 6.4.1 *Let us consider point groups in $\mathcal{O}(\mathbb{R}^2)$. Let P be an $\mathcal{S}$-symmetric point set in $\mathbb{R}^2$. See Figure 6.4 for examples of $\mathcal{C}_k$-symmetric infinitesimal isometries. In general, if $|P| > 1$,*

$$\dim \mathrm{iso}_{\mathcal{C}_k}(P) = \begin{cases} 3 & \text{if } k = 1 \\ 1 & \text{if } k \geq 2, \end{cases}$$

and if $P = \{x\}$,

$$\dim \mathrm{iso}_{\mathcal{C}_k}(P) = \begin{cases} 2 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \text{ (where } x \text{ should be the origin)} \end{cases}$$



(a) (b) (c)

Figure 6.4: Three independent infinitesimal isometries in the plane, among which (a) is symmetric with respect to the group of a vertical reflection, (b) is symmetric with respect to the group of a horizontal reflection, and (c) is symmetric with respect to the group of rotations.

Similarly, for the dihedral group $\mathcal{D}_k$ of order $2k$,

$$\dim \mathrm{iso}_{\mathcal{D}_k}(P) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2, \end{cases}$$

A result of Schulze [48] motivates us to look at $\mathcal{S}$-symmetric infinitesimal rigidity, which states that if (H, p) is not symmetry-forced infinitesimally rigid on an $\mathcal{S}$-generic p, then (H, p) has a nontrivial continuous motion that preserves the $(\mathcal{S}, \rho)$-symmetry.

6.4.3 The orbit rigidity matrix

Let (H, p) be an $(\mathcal{S}, \rho)$-symmetric framework in $\mathbb{R}^d$. Due to (6.5), the system

$$\langle m(u) - m(v), p(u) - p(v) \rangle = 0 \qquad \text{for all } \{u, v\} \in E(H) \tag{6.6}$$

of linear equations (with respect to m) is redundant. Schulze and Whiteley [51] pointed out that the system can be reduced to $|E(H)/\mathcal{S}|$ linear equations.

To see this, we first define a *joint-configuration* $\tilde{p}$ of vertex orbits by $\tilde{p} : V(H)/\mathcal{S} \to \mathbb{R}^d$. By taking a representative vertex v from each vertex orbit $\mathcal{S}v$, we set $\tilde{p}(\mathcal{S}v) = p(v)$. (Then, the locations of the other non-representative vertices are uniquely determined by (1.4.1).)

In a similar way, we define an *infinitesimal motion* of $(H/\mathcal{S}, \tilde{p})$ by $\tilde{m} : V(H)/\mathcal{S} \to \mathbb{R}^d$. By using the representative vertices determined above, we fix a one-to-one correspondence between $\mathcal{S}$-symmetric infinitesimal motions of $V(H)$ and infinitesimal motions of $V(H)/\mathcal{S}$ by $\tilde{m}(\mathcal{S}v) = m(v)$ for each vertex orbit $\mathcal{S}v$.

Let $(H/\mathcal{S}, \phi)$ be the quotient $\mathcal{S}$-gain graph of H. Recall that each (oriented) edge orbit $\mathcal{S}e$ connecting $\mathcal{S}u$ and $\mathcal{S}v$ with gain h_e can be written by $\mathcal{S}e = \{\{gu, gh_ev\} : g \in$

$\mathcal{S}\}$. The system (6.6) is hence written by

$$\langle m(gu) - m(gh_e v), p(gu) - p(gh_e v)\rangle = 0 \qquad \text{for all } \{gu, gh_e v\} \in \mathcal{S}e \tag{6.7}$$

over all edge orbits $\mathcal{S}e \in E(H)/\mathcal{S}$. Recall that the transpose of g is g^{-1} for any $g \in \mathcal{O}(\mathbb{R}^d)$. By (1.4.1) and (6.5),

$$\begin{aligned}\langle m(gu) - m(gh_e v), p(gu) - p(gh_e v)\rangle =& \langle m(u) - h_e m(v), p(u) - h_e p(v)\rangle \\ =& \langle m(u), p(u) - h_e p(v)\rangle + \langle m(v), p(v) - h_e^{-1} p(u)\rangle \\ =& \langle \tilde{m}(\mathcal{S}u), \tilde{p}(\mathcal{S}u) - h_e \tilde{p}(\mathcal{S}v)\rangle + \langle \tilde{m}(\mathcal{S}v), \tilde{p}(\mathcal{S}v) - h_e^{-1}\tilde{p}(\mathcal{S}u)\rangle.\end{aligned}$$

Therefore, for $\tilde{p} : V(H)/\mathcal{S} \to \mathbb{R}^d$, a mapping $\tilde{m} : H/\mathcal{S} \to \mathbb{R}^d$ is an infinitesimal motion of $(H/\mathcal{S}, \tilde{p})$ if and only if

$$\langle \tilde{m}(\mathcal{S}u), \tilde{p}(\mathcal{S}u) - h_e \tilde{p}(\mathcal{S}v)\rangle + \langle \tilde{m}(\mathcal{S}v), \tilde{p}(\mathcal{S}v) - h_e^{-1}\tilde{p}(\mathcal{S}u)\rangle = 0 \tag{6.8}$$

for every oriented edge orbit $\mathcal{S}e$ with $\phi(\mathcal{S}e) = h_e$. By regarding (6.8) as a system of linear equations of $\tilde{m}$, the corresponding $|E(H)/\mathcal{S}| \times d|V(H)/\mathcal{S}|$-matrix is called the *orbit rigidity matrix*.

In general, for an $\mathcal{S}$-gain graph (G, ϕ) and $\tilde{p} : V \to \mathbb{R}^d$, we shall define the *orbit rigidity matrix* as an $|E(G)| \times d|V(G)|$-matrix, in which each row corresponds to an edge, each vertex is associated with a d-tuple of columns, and the row corresponding to $e = (u, v) \in E(G)$ is written by

$$0 \dots 0 \quad \overbrace{\tilde{p}(u) - \phi(e)\tilde{p}(v)}^{u} \quad 0 \dots 0 \quad \overbrace{\tilde{p}(v) - \phi(e)^{-1}\tilde{p}(u)}^{v} \quad 0 \dots 0$$

if e is not a loop, and by

$$0 \dots 0 \quad \overbrace{(2I_d - \phi(e) - \phi(e)^{-1})\tilde{p}(v)}^{v} \quad 0 \dots 0$$

if e is a loop. The orbit rigidity matrix of $(G, \phi, \tilde{p})$ is denoted by $O(G, \phi, \tilde{p})$. From the above discussion, it follows that the dimension of the $\mathcal{S}$-symmetric infinitesimal motions can be computed from the rank of the orbit rigidity matrix of the corresponding quotient gain graph, which is formally stated as follows:

Theorem 6.4.2 (Schulze and Whiteley [51]) *Let (H, p) be an $(\mathcal{S}, \rho)$-symmetric framework with a free action ρ. Then,*

$$\dim L_{\mathcal{S}}(H, p) = d|V(H)/\mathcal{S}| - \operatorname{rank} O(H/\mathcal{S}, \phi, \tilde{p}),$$

where $(H/\mathcal{S}, \phi)$ is the quotient $\mathcal{S}$-gain graph and $\tilde{p}$ is a joint-configuration of vertex orbits corresponding to p.

6.4.4 Necessary condition for symmetric infinitesimal rigidity

We can show a necessary condition for the row independence of orbit rigidity matrices.

Lemma 6.4.3 *Let (G, ϕ) be an $\mathcal{S}$-gain graph with underlying graph $G = (V, E)$, and let $p : V \to \mathbb{R}^d$. If $O(G, \phi, p)$ is row independent, then*

$$|F| \leq \sum_{F_i \in C(F)} \{d|V(F_i)| - \dim \mathrm{iso}_{\langle F_i \rangle_{\phi,w}}(p(F_i))\}$$

for all $F \subseteq E$ and $w \in V(F_i)$, where $p(F_i) = \{gp(v) \colon v \in V(F_i), g \in \mathcal{S}\}$.

This, together with Theorem 6.4.2, directly implies a necessary condition for symmetric frameworks to be symmetry-forced infinitesimally rigid.

Recall that $\mathcal{S}$ is a finite family of orthogonal matrices. Let $\mathbb{Q}_{\mathcal{S}}$ be the field generated by $\mathbb{Q}$ and the entries of all the matrices in $\mathcal{S}$. Since $\mathcal{S}$ is finite, almost all numbers in $\mathbb{R}$ are transcendental over $\mathbb{Q}_{\mathcal{S}}$. For a given gain graph (G, ϕ), a mapping $\tilde{p} : V(G) \to \mathbb{R}^d$ is called *$\mathcal{S}$-generic* if the set of coordinates of $\tilde{p}(v)$ for all $v \in V(G)$ is algebraically independent over $\mathbb{Q}_{\mathcal{S}}$. Similarly, for a given $(\mathcal{S}, \rho)$-symmetric graph H, an $(\mathcal{S}, \rho)$-symmetric joint-configuration $p : V(H) \to \mathbb{R}^d$ is called *$\mathcal{S}$-generic* if the corresponding joint-configuration $\tilde{p}$ of the vertex orbits is $\mathcal{S}$-generic. An $\mathcal{S}$-symmetric framework is called *$\mathcal{S}$-generic* if the joint configuration is $\mathcal{S}$-generic.

We will check that the condition of Lemma 6.4.3 is indeed sufficient for generic symmetric frameworks in the plane with cyclic groups and dihedral groups $\mathcal{D}_k$ with odd k, respectively.

6.5 Characterization of symmetry-forced rigid graphs for cyclic point groups

The following lemma is a key observation, which is an extension of the one given in [60, 68] for proving Laman's theorem. The lemma is not limited to cyclic groups.

Lemma 6.5.1 *Let (G, ϕ) be an $\mathcal{S}$-gain graph for a point group $\mathcal{S} \subset \mathcal{O}(\mathbb{R}^2)$. Let (G', ϕ') be an $\mathcal{S}$-gain graph obtained from (G, ϕ) by a 0-extension, 1-extension, or loop-1-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

Proof. If there is a p such that $O(G, \phi, p)$ is row independent, then $O(G, \phi, q)$ is row independent for all $\mathcal{S}$-generic q. Hence, we may assume that p is $\mathcal{S}$-generic. We only

show the difficult case where (G', ϕ') is constructed from (G, ϕ) by a 1-extension. (See [27] for the easier case where (G', ϕ') is constructed from (G, ϕ) by a 0-extension or a loop-1-extension.)

Suppose that (G', ϕ') is obtained from (G, ϕ) by a 1-extension, by removing an existing edge e and adding a new vertex v with three new non-loop edges e_1, e_2, e_3 incident to v. We may assume that e_i is outgoing from v. Let u_i be the other endvertex of e_i, and let $g_i = \phi'(e_i)$ and $p_i = p(u_i)$ for $i = 1, 2, 3$. By the definition of 1-extension, we have $\phi(e) = g_1^{-1} g_2$.

Claim 6.5.2 *The three points $g_i p_i$ $(i = 1, 2, 3)$ do not lie on a line.*

Proof. Since p is $\mathcal{S}$-generic, $u_1 = u_2 = u_3$ should hold if they lie on a line. Then $p_1 = p_2 = p_3$. By the definition of 1-extensions, $g_i \neq g_j$ if $u_i = u_j$. This implies that $g_1 p_1, g_2 p_2, g_3 p_3$ are three distinct points on a circle. Thus, they do not lie on a line. $\square$

We take $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for all $w \in V(G)$, and $p'(v)$ is a point on the line through $g_1 p_1$ and $g_2 p_2$ but neither $g_1 p_1$ nor $g_2 p_2$. $O(G', \phi', p')$ is described as follows: if $u_1 \neq u_2$

	v	u_1	u_2	
e_3	$p'(v) - g_3 p_3$	$*$	$*$	$*$
e_1	$p'(v) - g_1 p_1$	$p_1 - g_1^{-1} p'(v)$	0	0
e_2	$p'(v) - g_2 p_2$	0	$p_2 - g_2^{-1} p'(v)$	0
$E(G) - e$	0	$O(G - e, \phi, p)$		

where the right-bottom block $O(G - e, \phi, p)$ denotes the orbit rigidity matrix obtained from $O(G, \phi, p)$ by removing the row of e, whereas, if $u_1 = u_2$,

	v	u_1	
e_3	$p'(v) - g_3 p_3$	$*$	$*$
e_1	$p'(v) - g_1 p_1$	$p_1 - g_1^{-1} p'(v)$	0
e_2	$p'(v) - g_2 p_1$	$p_1 - g_2^{-1} p'(v)$	0
$E(G) - e$	0	$O(G - e, \phi, p)$	

We consider the case when $u_1 \neq u_2$. (The case when $u_1 = u_2$ is similar.) Since $p'(v)$ lies on the line through $g_1 p_1$ and $g_2 p_2$, $p'(v) - g_i p(u_i)$ is a scalar multiple of $g_1 p_1 - g_2 p_2$ for $i = 1, 2$. Hence, by multiplying the rows of e_1 and e_2 by an appropriate

scalar, $O(G', \phi', p')$ becomes

	v	u_1	u_2	
e_3	$p'(v) - g_3p_3$	$*$	$*$	$*$
e_1	$g_1p_1 - g_2p_2$	$-g_1^{-1}(g_1p_1 - g_2p_2)$	0	0
e_2	$g_1p_1 - g_2p_2$	0	$-g_2^{-1}(g_1p_1 - g_2p_2)$	0
$E(G) - e$	0	$O(G - e, \phi, p)$		

Subtracting the row of e_1 from that of e_2, we finally get

	v	u_1	u_2	
e_3	$p'(v) - g_3p_3$	$*$	$*$	$*$
e_1	$g_1p_1 - g_2p_2$	$-g_1^{-1}(g_1p_1 - g_2p_2)$	0	0
e_2	0	$p_1 - g_1^{-1}g_2p_2$	$p_2 - g_2^{-1}g_1p_1$	0
$E(G) - e$	0	$O(G - e, \phi, p)$		

Since $\phi(e) = g_1^{-1}g_2$, the row of e_2 is equal to the row of e in $O(G, \phi, p)$. This means that the right-bottom block together with the row of e_2 forms $O(G, \phi, p)$, which is row independent. Thus, the matrix is row independent if and only if the top-left block is row independent. Since g_ip_i $(i = 1, 2, 3)$ are not on a line by Claim 6.5.2, the line through $p'(v)$ and g_3p_3 is not parallel to the line through g_1p_1 and g_2p_2. This implies that the top-left block is row independent, and consequently $O(G', \phi', p')$ is row independent. □

We are now ready to prove a combinatorial characterization of the infinitesimal rigidity of $\mathcal{S}$-generic symmetric frameworks with cyclic point groups in the plane. The same statement was also proved in [39] for rotation group and in [38] for reflection group by completely different proofs.

Theorem 6.5.3 *Let $\mathcal{C} \subset \mathcal{O}(\mathbb{R}^2)$ be a cyclic point group, that is, either a group of k-fold rotations or a group of a reflection, and let (H, p) be a generic $(\mathcal{C}, \rho)$-symmetric framework in the plane with a free action ρ. Then (H, p) is symmetry-forced infinitesimally rigid if and only if the quotient $\mathcal{C}$-gain graph contains a spanning maximum $(2, 3)$-g-tight subgraph.*

Proof. By Theorem 6.4.2 it suffices to show that for the quotient $\mathcal{C}$-gain graph $(H/\mathcal{C}, \phi)$ and any $\mathcal{C}$-generic $\tilde{p} : V(H/\mathcal{C}) \to \mathbb{R}^2$, $O(H/\mathcal{C}, \phi, \tilde{p})$ is row independent if and only if $(H/\mathcal{C}, \phi)$ is $(2, 3)$-g-sparse. Let us simply denote $G = H/\mathcal{C}$.

("If part") It suffices to consider the case when G is maximum $(2, 3)$-g-tight. The proof is done by induction on $|V(G)|$. For $|V(G)| = 1$, G consists of single vertex with an unbalanced loop. Then $O(G, \phi, \tilde{p})$ consists of a nonzero row, which implies that $O(G, \phi, \tilde{p})$ is row-independent.

For $|V(G)| > 1$, by Theorem 6.3.4, G can be built up from a $\mathcal{C}$-gain graph with one vertex and an unbalanced loop with a sequence of 0-extensions, 1-extensions, and loop-1-extensions. Thus, there is a maximum $(2,3)$-g-tight graph (G', ϕ') from which (G, ϕ) is constructed by a 0-extension, 1-extension, or loop-1-extension. By induction, there is a p' such that $O(G, \phi', p')$ is row independent. Thus, Lemma 6.5.1 implies that there is a p such that $O(G, \phi, p)$ is row independent, which in turn implies that $O(G, \phi, q)$ is row independent for all $\mathcal{C}$-generic q.

("Only-if part") The necessity is based on Lemma 6.4.3. Suppose that $O(G, \phi, \tilde{p})$ is row independent. Recall that we have seen the exact value of $\dim \mathrm{iso}_{\mathcal{C}}(P)$ for $\mathcal{C} \subset \mathcal{O}(\mathbb{R}^2)$ and a $\mathcal{C}$-symmetric point set $P \subseteq \mathbb{R}^2$ in Example 6.4.1. Since $\tilde{p}$ is $\mathcal{C}$-generic, we have

$$\mathrm{iso}_{\langle F \rangle_v}(\tilde{p}(F)) = \begin{cases} 3 & \text{(if } F \text{ is balanced)} \\ 1 & \text{(otherwise)} \end{cases}$$

for all connected $F \subseteq E(G)$ and $v \in V(F)$, where $\tilde{p}(F) = \{g\tilde{p}(v) \colon v \in V(F), g \in \mathcal{C}\}$. Therefore, by Lemma 6.4.3, we have

$$|F| \leq \sum_{F' \in C(F)} \{2|V(F')| - \mathrm{iso}_{\langle F' \rangle_v}(\tilde{p}(F'))\} \leq 2|V(F)| - \begin{cases} 3 & \text{(if } F \text{ is balanced)} \\ 1 & \text{(otherwise)} \end{cases}$$

for all $F \subseteq E(G)$. Therefore, (G, ϕ) is $(2,3)$-g-sparse. □

6.6 Combinatorial characterization of symmetry-forced rigidity with $\mathcal{D}_k$-symmetry for odd k

In the previous sections we gave a constructive characterization of $(2,3)$-g-sparse graphs and their realizations as symmetry-forced rigid frameworks in the plane with cyclic point group symmetry. We next move to non-cyclic point groups, that is, dihedral groups of order $2k$ that we denote by $\mathcal{D}_k$ (or simply by $\mathcal{D}$). The corresponding matroid, that we construct in the next subsection, is slightly different from the $(2,3)$-g-count matroid, as we need to take into account the fact that the underlying group is not cyclic.

6.6.1 $\mathcal{D}$-sparsity

Let (G, ϕ) be a $\mathcal{D}$-gain graph with underlying graph $G = (V, E)$. We define a function $f_{\mathcal{D}} : 2^E \to \mathbb{Z}$ by

$$f_{\mathcal{D}}(X) = 2|V(X)| - 3 + \beta(X) \qquad (X \subseteq E)$$

where

$$\beta(X) = \begin{cases} 0 & \text{(if } X \text{ is balanced)} \\ 2 & \text{(if } X \text{ is unbalanced and cyclic)} \\ 3 & \text{(otherwise),} \end{cases}$$

and define a class of sparse graphs determined by $f_{\mathcal{D}}$ as follows.

Definition 6.6.1 *Let (G, ϕ) be a $\mathcal{D}$-gain graph. An edge set $X \subseteq E(G)$ is called* $\mathcal{D}$-sparse *if $|F| \leq f_{\mathcal{D}}(F)$ for any nonempty $F \subseteq X$, and it is called* $\mathcal{D}$-tight *if it is $\mathcal{D}$-sparse with $|X| = f_{\mathcal{D}}(X)$.*

(G, ϕ) is said to be $\mathcal{D}$-sparse *if so is $E(G)$, and it is called* maximum $\mathcal{D}$-tight *if it is $\mathcal{D}$-sparse with $|E(G)| = 2|V(G)|$.*

By a simple degree of freedom counting argument based on Example 6.4.1 and Lemma 6.4.3, it is not difficult to see that the $\mathcal{D}$-sparsity is a necessary condition for orbit rigidity matrices to be row independent in case of dihedral symmetry. (A formal proof will be given in Lemma 6.6.12.) To prove the sufficiency, the first question is whether $\mathcal{D}$-sparsity defines a collection of independent sets of a matroid. This will be proved in this subsection.

We will use the following technical lemmas on properties of $\mathcal{D}$-tight sets.

Lemma 6.6.2 *Let (G, ϕ) be a $\mathcal{D}$-sparse graph with $G = (V, E)$ and $F \subseteq E$ be a $\mathcal{D}$-tight set. Then, the following holds.*

(i) *If F is cyclic, then F is connected.*

(ii) *If F is balanced with $|F| > 1$, then F has neither parallel edges nor loops and is 2-connected and essentially 3-edge-connected.*

Proof. Since G is $\mathcal{D}$-sparse and β is monotone nondecreasing, we have $|F| \leq \sum_{F' \in C(F)} f_{\mathcal{D}}(F') \leq 2|V(F)| - (3 - \beta(F))c$, where c denotes the number of connected components in F. Hence, if F is not connected and $\beta(F) < 3$, then $|F| < 2|V(F)| - 3 + \beta(F)$, implying that F is not $\mathcal{D}$-tight. Therefore if $\beta(F) < 3$ then X is connected.

Suppose further that F is balanced. Then we have $\beta(X) = 0$ for any $X \subseteq F$. This means that $|X| \leq f_{2,3}(X)$ for any nonempty $X \subseteq F$, and $|F| = f_{\mathcal{D}}(F) = 2|V(F)| - 3 = f_{2,3}(F)$. In other words, F is independent in the generic 2-rigidity matroid $\mathcal{M}(f_{2,3})$ of $G[F]$. It is known that, in the generic 2-rigidity matroid, an independent set F with $|F| = f_{2,3}(F)$ and $|F| > 1$ has neither parallel edges nor a loop and is 2-connected and essentially 3-edge-connected (see e.g. [21]). $\square$

Lemma 6.6.3 *Let (G, ϕ) be a $\mathcal{D}$-sparse graph with $G = (V, E)$. Let $X, Y \subseteq E$ be $\mathcal{D}$-tight edge sets with $X \cap Y \neq \emptyset$. Then $X \cup Y$ is $\mathcal{D}$-tight.*

Proof. Without loss of generality, assume $\beta(X) \geq \beta(Y)$.

Let $d = 2|V(X \cup Y)| - |X \cup Y|$. Note that $X \cup Y$ is $\mathcal{D}$-tight if one of the following holds: (i) $d = 0$, (ii) $d \leq 1$ and $X \cup Y$ is cyclic, or (iii) $d \leq 3$ and $X \cup Y$ is balanced.

Let c_0 be the number of isolated vertices in the graph $(V(X) \cap V(Y), X \cap Y)$ and c_1 be the number of connected components in $X \cap Y$. We have $|X| = 2|V(X)| - 3 + \beta(X)$ and $|Y| = 2|V(Y)| - 3 + \beta(Y)$. We also have

$$\begin{aligned} |X \cap Y| &\leq \sum_{F \in C(X \cap Y)} f_{\mathcal{D}}(F) = 2|V(X \cap Y)| - 3c_1 + \sum_{F \in C(X \cap Y)} \beta(F) \\ &= 2|V(X) \cap V(Y)| - 2c_0 - 3c_1 + \sum_{F \in C(X \cap Y)} \beta(F) \\ &\leq 2|V(X) \cap V(Y)| - 2c_0 - 3c_1 + \beta(Y)c_1 \end{aligned} \tag{6.9}$$

since β is monotone non-decreasing. Therefore,

$$\begin{aligned} d &= 2|V(X \cup Y)| - |X \cup Y| = 2|V(X \cup Y)| - (|X| + |Y| - |X \cap Y|) \\ &\leq 6 - \beta(X) - \beta(Y) - 2c_0 - 3c_1 + \beta(Y)c_1 \\ &\leq 3 - \beta(X) - 2c_0 - (3 - \beta(Y))(c_1 - 1). \end{aligned} \tag{6.10}$$

Note that $c_1 \geq 1$ by $X \cap Y \neq \emptyset$ and hence $(3 - \beta(Y))(c_1 - 1) \geq 0$.

If $\beta(X) = 3$, then (6.10) implies that $d = 0$ and hence $X \cup Y$ is $\mathcal{D}$-tight.

Therefore we assume $\beta(X) < 3$. Then X and Y are connected by Lemma 6.6.2. We split the proof into two cases depending on the value of $\beta(X)$.

(Case 1) If $\beta(X) = 2$, then (6.10) implies that $d \leq 1$. Since $d = 0$ implies the $\mathcal{D}$-tightness of $X \cup Y$, let us assume $d = 1$ and prove that $X \cup Y$ is cyclic. If $d = 1$, then the inequalities of (6.9) and (6.10) hold with equalities, and in particular $c_0 = 0$, $c_1 = 1$ and

$$|X \cap Y| = 2|V(X \cap Y)| - 3 + \beta(Y). \tag{6.11}$$

By $c_0 = 0$ and $c_1 = 1$, the number of connected components in the graph $(V(X) \cap V(Y), X \cap Y)$ is one. If $\beta(Y) = 2$, then $X \cap Y$ is unbalanced cyclic by (6.11) and hence Lemma 6.1.4(3) implies that $X \cup Y$ is cyclic. If $\beta(Y) = 0$, then Y is balanced and, again, Lemma 6.1.4(2) implies that $X \cup Y$ is cyclic. Thus $X \cup Y$ is $\mathcal{D}$-tight.

(Case 2) If $\beta(X) = 0$, then $\beta(Y) = 0$ and we have $d \leq 6 - 2c_0 - 3c_1$ by (6.10). By $c_1 \geq 1$, we have three possible pairs $(c_0, c_1) = (0, 1), (1, 1), (0, 2)$. If $(c_0, c_1) = (0, 1)$, then $d \leq 3$ and Lemma 6.1.4 implies that $X \cup Y$ is balanced. Thus, $X \cup Y$ is a balanced $\mathcal{D}$-tight set. If $(c_0, c_1) = (1, 1)$ or $(c_0, c_1) = (0, 2)$, then $d \leq 1$ and Lemma 6.1.5 implies that $X \cup Y$ is cyclic. Thus, $X \cup Y$ is a cyclic $\mathcal{D}$-tight set.

This completes the proof. $\square$

Lemma 6.6.4 *Let (G, ϕ) be a $\mathcal{D}$-gain graph with $G = (V, E)$ and X and Y be $\mathcal{D}$-tight sets with $X \subseteq Y \subseteq E$. For $e \in E \setminus Y$, if $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X + e)$, then $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y + e)$.*

Proof. Since $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X + e)$, the endvertices of e are contained in $V(X)$, implying $V(Y+e) = V(Y)$. If X or Y is not cyclic, then we have $\beta(Y) = \beta(Y+e) = 3$, meaning that $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y + e)$.

We hence assume that X and Y are cyclic, and they are connected by Lemma 6.6.2. Take a spanning tree T in $G[Y]$ such that $X \cap T$ is a spanning tree of $G[X]$. By Proposition 6.1.2, there is an equivalent gain function ϕ' to ϕ such that $\phi'(f) = \mathsf{id}$ for $f \in T$. By Lemma 6.1.3, there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi'(f) \in \mathcal{C}$ for every $f \in Y$, where $\mathcal{C}$ is the trivial group if Y is balanced. Since $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X+e)$ and $X \subseteq Y$, we have $\phi'(e) \in \bar{\mathcal{C}}$, and hence $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y + e)$ holds. $\square$

We are ready to prove that the family of $\mathcal{D}$-sparse edge subsets is a family of independent sets of a matroid on ground-set E. We shall also characterize the rank function of this matroid.

Theorem 6.6.5 *Let (G, ϕ) be a $\mathcal{D}$-gain graph with $G = (V, E)$ and $\mathcal{I}$ be the family of all $\mathcal{D}$-sparse edge subsets in E. Then $\mathcal{M}_{\mathcal{D}}(G, \phi) = (E, \mathcal{I})$ is a matroid on ground-set E. The rank of a set $E' \subseteq E$ in $\mathcal{M}_{\mathcal{D}}(G, \phi)$ is equal to*

$$\min \left\{ \sum_{i=1}^{t} f_{\mathcal{D}}(E'_i) \colon \{E'_1, \ldots, E'_t\} \ \textit{ is a partition of } E' \right\}.$$

Proof. For a partition $\mathcal{P} = \{E'_1, \ldots, E'_t\}$ of $E' \subseteq E$, we denote $\mathrm{val}(\mathcal{P}) = \sum_{i=1}^{t} f_{\mathcal{D}}(E'_t)$. We shall check the following independence axiom of matroids: (I1) $\emptyset \in \mathcal{I}$; (I2) for any $X, Y \subseteq E$ with $X \subseteq Y$, $Y \in \mathcal{I}$ implies $X \in \mathcal{I}$; (I3) for any $E' \subseteq E$, maximal subsets of E' belonging to $\mathcal{I}$ have the same cardinality.

$\mathcal{I}$ obviously satisfies (I1) and (I2). To see (I3), let $E' \subseteq E$ and let $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \mathrm{val}(\mathcal{P})$ for all partitions $\mathcal{P}$ of E'. We shall prove that there is a partition $\mathcal{P}$ of E' with $|F| = \mathrm{val}(\mathcal{P})$, from which (I3) follows.

Let $J = (V, F)$ denote the subgraph with the edge set F. Consider the family $\{F_1, F_2, \ldots, F_t\}$ of all maximal $\mathcal{D}$-tight sets in J. Since each edge $f \in F$ forms a $\mathcal{D}$-tight set, $\cup_{i=1}^{t} F_i = F$ holds. Since $F_i \cap F_j = \emptyset$ for every pair $1 \leq i < j \leq t$ by Lemma 6.6.3 and the maximality, $\mathcal{P}_F = \{F_1, F_2, \ldots, F_t\}$ is a partition of F and $|F| = \mathrm{val}(\mathcal{P}_F)$ follows.

Based on $\mathcal{P}_F$, we construct a partition $\mathcal{P}$ of E' with $\mathrm{val}(\mathcal{P}) = \mathrm{val}(\mathcal{P}_F) = |F|$. Consider an edge $(u, v) = e \in E' - F$. Since F is a maximal subset of E' in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. Hence there must be a tight set X_e in J with $u, v \in V(X_e)$ and $X_e + e \notin \mathcal{I}$. $X_e \subseteq F_i$ for some $1 \leq i \leq t$. Choose such an F_i for every $e \in E' - F$ and define $E_i = F_i \cup \{e : F_i \text{ was chosen for } e\}$. Clearly $\mathcal{P} = \{E_1, E_2, \ldots, E_t\}$ is a partition of E'. By Lemma 6.6.4, $f_{\mathcal{D}}(F_i) = f_{\mathcal{D}}(E_i)$ for every $1 \leq i \leq t$ and hence $\mathrm{val}(\mathcal{P}) = \mathrm{val}(\mathcal{P}_F) = |F|$. □

The matroid which was introduced and denoted by $\mathcal{M}_{\mathcal{D}}(G, \phi)$ in Theorem 6.6.5 is called the *$\mathcal{D}$-sparsity matroid* of (G, ϕ).

6.6.2 Constructive characterization of maximum $\mathcal{D}$-tight graphs

We now present a constructive characterization of maximum $\mathcal{D}$-tight graphs. Notice that the average vertex degree in a maximum $\mathcal{D}$-tight graph (G, ϕ) is four, which means that G has a vertex of degree at most 3 if and only if G is not 4-regular. Thus we shall take a special care of 4-regular $\mathcal{D}$-sparse graphs.

0-extension, 1-extension, and loop-1-extension

Before looking at 4-regular graphs and vertices of degree four, we consider the 0-extension, 1-extension, and loop-1-extension operations. Recall that the corresponding inverse operations are called reductions. A reduction is *admissible* if the resulting graph is $\mathcal{D}$-sparse.

Lemma 6.6.6 *Let (G, ϕ) be a $\mathcal{D}$-sparse graph with $G = (V, E)$. Applying a 0-extension, 1-extension or loop-1-extension to G results in a $\mathcal{D}$-sparse graph with $|V| + 1$ vertices and $|E| + 2$ edges.*

Conversely, for any vertex v of degree 2 or 3, the 0-reduction, loop-1-reduction, or some of the 1-reductions at v is admissible if $|V| \geq 2$.

Proof. The proof of the first claim is exactly the same as the proof of Lemma 6.3.1. (Indeed, we just need to change $2\alpha(F)$ with $\beta(F)$ in the proof of Lemma 6.3.1.)

To see that some reduction is admissible at a vertex v of degree three, we just need to observe that each circuit of $\mathcal{M}(g_{2,3})$ appearing in the proof of Claim 6.3.3 is also a circuit in $\mathcal{M}_{\mathcal{D}}(G, \phi)$. We can thus apply exactly the same proof as in Lemma 6.3.2 to conclude that some reduction is admissible at v. □

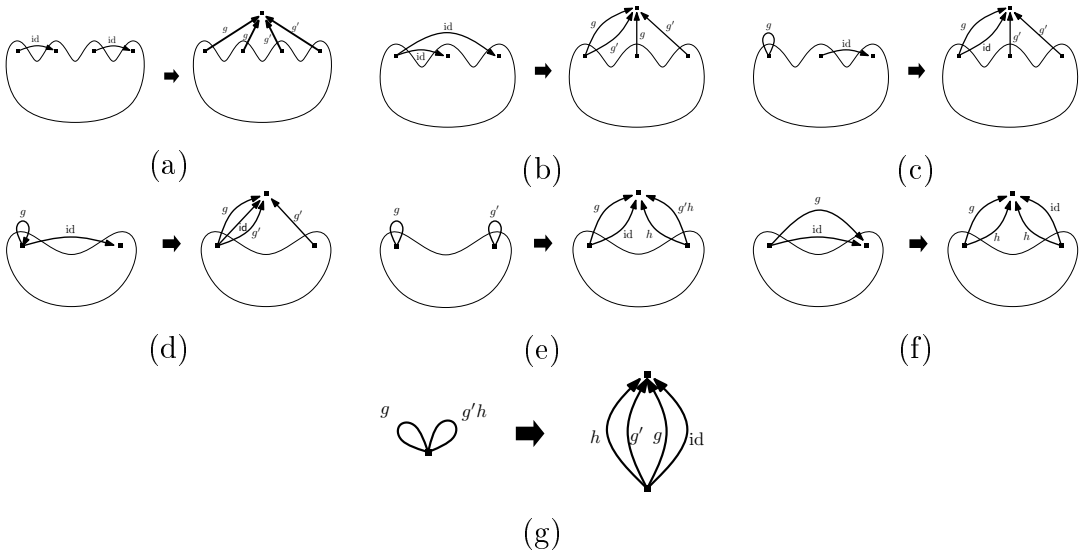


Figure 6.5: 2-extensions

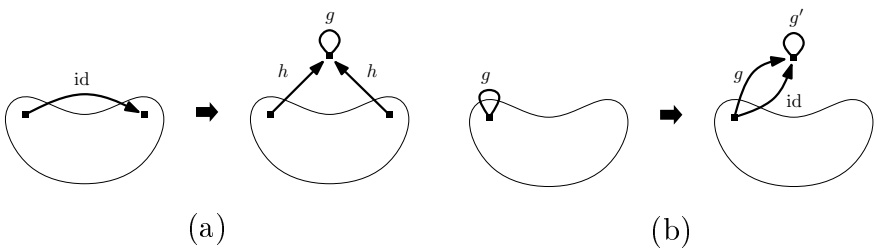


Figure 6.6: Loop-2-extensions.

2-extension and loop-2-extension

Besides 0-extensions, 1-extensions and loop-1-extensions, we shall introduce *2-extensions* and *loop-2-extensions* for constructing 4-regular $\mathcal{D}$-sparse graphs.

In a *2-extension*, we take two existing edges $e = (v_1, v_2)$ and $f = (v_3, v_4)$ and pinch them by inserting a new vertex v. More precisely, a 2-extension removes e and f, inserts a new vertex v with four new edges, e_i from v_i to v for each $i = 1, \ldots, 4$. The gain function ϕ is extended on $E \cup \{e_1, \ldots, e_4\}$ so that $\phi(e_1) \cdot \phi(e_2)^{-1} = \phi(e)$, $\phi(e_3) \cdot \phi(e_4)^{-1} = \phi(f)$ and it is locally $\mathcal{D}$-sparse, i.e., $\{e_1, \ldots, e_4\}$ is $\mathcal{D}$-sparse. Depending on the multiplicity of the v_i's we have seven cases as shown in Figure 6.5.

In a *loop-2-extension*, we remove an existing edge $e = (v_1, v_2)$, insert a new vertex v, a new loop l at v and two new edges, e_i from v_i to v for each $i = 1, 2$. ϕ is extended on $E \cup \{e_1, e_2, l\}$ so that $\phi(e_1) \cdot \phi(e_2)^{-1} = \phi(e)$, $\phi(l) \neq \mathrm{id}$ and it is locally $\mathcal{D}$-sparse. Depending on whether e is a loop or not, we have two cases as shown in Figure 6.6.

The following lemma shows that these operations preserve $\mathcal{D}$-sparsity.

Lemma 6.6.7 *Let (G, ϕ) be a $\mathcal{D}$-sparse graph. Then, any $\mathcal{D}$-gain graph (G', ϕ') obtained from G by a 2-extension or a loop-2-extension is $\mathcal{D}$-sparse.*

Proof. Suppose that (G', ϕ') is obtained by a 2-extension. Let us denote the removed edges by e and f and the new edges by $e_1, \ldots, e_4$ as above. Suppose that there is $F \subseteq E(G')$ that violates the $\mathcal{D}$-sparsity condition. Let $F' = F \setminus \{e_1, \ldots, e_4\}$. Since $\{e_1, \ldots, e_4\}$ satisfies the $\mathcal{D}$-sparsity condition, $F' \neq \emptyset$. Let us add e to F' if $\{e_1, e_2\} \subseteq F$ and add f to F' if $\{e_3, e_4\} \subseteq F$. Observe that $|F'| \geq |F| - 2$, $|V(F)| \geq |V(F')| + 1$ and $\beta(F) \geq \beta(F')$. Since $|F| > f_{\mathcal{D}}(F)$, we obtain $|F'| \geq |F| - 2 > f_{\mathcal{D}}(F) - 2 = 2|V(F)| - 3 + \beta(F) - 2 \geq 2|V(F')| - 3 + \beta(F') = f_{\mathcal{D}}(F')$. This contradicts the $\mathcal{D}$-sparsity of G since $\emptyset \neq F' \subseteq E(G)$. Therefore (G', ϕ') is $\mathcal{D}$-sparse.

In the same manner, it can be easily checked that a loop-2-extension also preserves $\mathcal{D}$-sparsity. □

We shall define the inverse moves of these operations. Recall that, for a vertex v and two incoming non-loop edges $e_1 = (u, v)$ and $e_2 = (w, v)$, we denote by $e_1 \cdot e_2^{-1}$ a new edge from u to w with gain $\phi(e_1) \cdot \phi(e_2)^{-1}$.

Let v be a vertex of degree four, not incident to a loop, and $e_i = (v_i, v)$ for $i = 1, \ldots, 4$ be the edges incident to v, assuming that all of them are oriented to v. The *2-reduction* (at v) deletes v and adds one of $\{e_1 \cdot e_2^{-1}, e_3 \cdot e_4^{-1}\}$, $\{e_1 \cdot e_3^{-1}, e_2 \cdot e_4^{-1}\}$ and $\{e_1 \cdot e_4^{-1}, e_2 \cdot e_3^{-1}\}$. We sometimes refer to a specific one: the *2-reduction at v through (e_i, e_j) and (e_k, e_l)* deletes v and adds $\{e_i \cdot e_j^{-1}, e_k \cdot e_l^{-1}\}$.

Let v be a vertex of degree four, incident to a loop l, and $e_i = (v_i, v)$ for $i = 1, 2$ be the non-loop edges incident to v, assuming that all of them are oriented to v. The *loop-2-reduction* (at v) deletes v and adds $e_1 \cdot e_2^{-1}$.

A 2-reduction or a loop-2-reduction is said to be *admissible* if the resulting graph is $\mathcal{D}$-sparse.

Base graphs

Our main theorem asserts that these operations are sufficient to construct all 4-regular $\mathcal{D}$-sparse graphs from certain classes of $\mathcal{D}$-sparse graphs. Here, the classes can be categorized into three groups: the first group includes special small graphs as in the conventional constructive characterizations, the second group is a class of graphs, which are obtained from cycles by duplicating each edge, and the third one consists of *near-cyclic 4-regular graphs*.

The first group consists of three types of special $\mathcal{D}$-tight graphs, called *trivial graphs*, *fancy triangles*, and *fancy hats*. A *trivial graph* is a $\mathcal{D}$-sparse graph with a single vertex and with two loops as shown in Figure 6.7(a). The gain function is assigned so that the gains of two loops generate a non-cyclic group.

A *fancy triangle* is a $\mathcal{D}$-gain graph whose underlying graph is obtained from

a triangle by adding a loop to each vertex, as shown in Figure 6.7(b). The gain function is assigned so that it is $\mathcal{D}$-sparse and the triangle is balanced.

A *hat* is a graph obtained from $K_{2,3}$ by adding an edge to the class of cardinality two, and the *fancy hat* is a $\mathcal{D}$-gain graph obtained from the hat by adding a loop to each degree two vertex, as shown in Figure 6.7(c). The gain function is assigned so that it is $\mathcal{D}$-sparse and the hat is balanced.

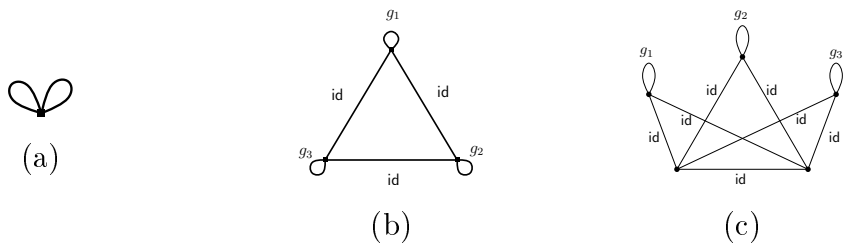


Figure 6.7: Special graphs: (a) a trivial graph, (b) a fancy triangle, and (c) a fancy hat.

The second group consists of $\mathcal{D}$-sparse graphs whose underlying graphs are double cycles, where, for $n \geq 2$, the *double cycle* C_n^2 is defined as the graph obtained from the cycle on n vertices by replacing each edge by two parallel edges as shown in Figure 6.8. As we will see later, key properties of this group depend on the parity of k of the underlying dihedral group $\mathcal{D}_k$.

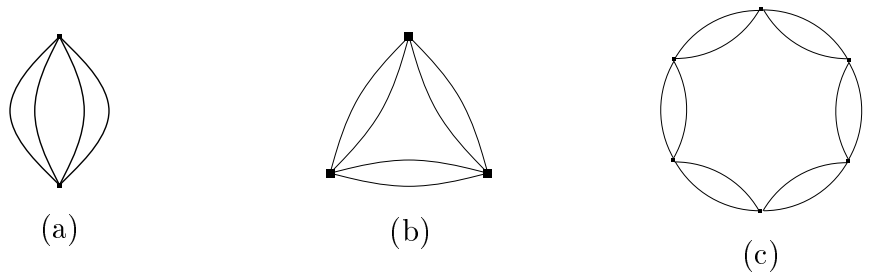


Figure 6.8: Double cycles: (a)C_2^2, (b)C_3^2, (c)C_6^2.

The third group consists of *near-cyclic* graphs, which, intuitively speaking, are the $\mathcal{D}$-tight graphs closest to $(2,3)$-g-tight graphs. More precisely, we say that a $\mathcal{D}$-sparse graph (G,ϕ) is *near-cyclic* if there is an edge e such that $(G-e,\phi)$ is cyclic.

The following lemma shows how to construct near-cyclic graphs.

Lemma 6.6.8 *Let (G,ϕ) be a $(2,3)$-g-sparse $\mathcal{D}$-gain graph with $G=(V,E)$, and suppose that there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi(e) \in \mathcal{C}$ for all $e \in E$. If we add a new edge e having a gain in $\mathcal{D} \setminus \bar{\mathcal{C}}$, then $(G+e,\phi)$ is $\mathcal{D}$-sparse.*

Constructive characterizations

We are ready to state our constructive characterization of 4-regular $\mathcal{D}$-sparse graphs. We say that a 4-regular $\mathcal{D}$-sparse graph is a *base graph* if it is a trivial graph, a fancy triangle, a fancy hat, or a near-cyclic graph.

Theorem 6.6.9 *Let (G, ϕ) be a $\mathcal{D}$-gain graph. Then, (G, ϕ) is 4-regular and $\mathcal{D}$-sparse if and only if it can be built up from a disjoint union of base graphs and $\mathcal{D}$-sparse double cycles by a sequence of 2-extension and loop-2-extension operations.*

Combining Theorem 6.6.9 and Lemma 6.6.6, we obtain the following:

Theorem 6.6.10 *Let (G, ϕ) be a $\mathcal{D}$-gain graph. Then, (G, ϕ) is maximum $\mathcal{D}$-tight if and only if it can be built up from a disjoint union of base graphs and $\mathcal{D}$-sparse double cycles by a sequence of 0-extension, 1-extension, loop-1-extension, 2-extension and loop-2-extension operations.*

The theorems can be strengthened if k is odd, in which case every $\mathcal{D}_k$-sparse double cycle can be reduced to a trivial graph.

Theorem 6.6.11 *Let $\mathcal{D}_k$ be a dihedral group with odd k. Then a $\mathcal{D}_k$-gain graph (G, ϕ) is maximum $\mathcal{D}_k$-tight if and only if it can be built up from a disjoint union of base graphs by a sequence of 0-extension, 1-extension, loop-1-extension, 2-extension and loop-2-extension operations.*

6.6.3 The characterization

In this section we discuss our combinatorial characterization of symmetry-forced infinitesimal rigidity with dihedral symmetry. We begin with a necessary condition based on Lemma 6.4.3.

Lemma 6.6.12 *Let $\mathcal{D}_k$ be a dihedral group with $k \geq 2$, and (H, p) be a generic $(\mathcal{D}_k, \rho)$-symmetric framework with a free action ρ. If (H, p) is symmetry-forced infinitesimally rigid, then the quotient gain graph contains a spanning maximum $\mathcal{D}_k$-tight subgraph.*

Theorem 6.6.13 *Let $\mathcal{D}_k$ be a dihedral group with odd $k \geq 3$, and (H, p) be a generic $(\mathcal{D}_k, \rho)$-symmetric framework with a free action ρ. Then (H, p) is symmetry-forced infinitesimally rigid if and only if the quotient gain graph contains a spanning maximum $\mathcal{D}_k$-tight subgraph.*

Necessity follows from Lemma 6.6.12. Therefore, by Theorem 6.4.2, it suffices to prove that, for a maximum $\mathcal{D}_k$-tight graph (G, ϕ), there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent.

By Theorem 6.6.11, (G, ϕ) can be constructed from a disjoint union of base graphs by 0-extension, 1-extension, loop-1-extension, 2-extension, and loop-2-extension operations. Therefore, what we have to prove is that (i) the orbit rigidity matrix of each base graph is row independent and (ii) each extension preserves the row independence of the orbit rigidity matrix by extending p appropriately. (i) will be solved in Lemma 6.6.14 whereas (ii) will be solved in Lemmas 6.6.15 and 6.6.16. Note that there is no parity condition in these lemmas.

Lemma 6.6.14 *Let (G, ϕ) be a base graph. Then, there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent.*

The next two lemmas show that loop-2-extensions and 2-extensions preserve the independence of rigidity matrices.

Lemma 6.6.15 *Let (G, ϕ) be a maximum $\mathcal{D}_k$-tight graph with $k \geq 2$ and (G', ϕ') a maximum $\mathcal{D}_k$-tight graph obtained from (G, ϕ) by a loop-2-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

Lemma 6.6.16 *Let (G, ϕ) be a maximum $\mathcal{D}_k$-tight graph with $k \geq 2$ and (G', ϕ') a maximum $\mathcal{D}_k$-tight graph obtained from (G, ϕ) by a 2-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

Combining Theorem 6.6.11, Lemma 6.5.1, Lemma 6.6.12, Lemma 6.6.14, Lemma 6.6.15, and Lemma 6.6.16, we can now complete the proof of Theorem 6.6.13.

Chapter 7

Lifting symmetric pictures to polyhedral scenes

Scene analysis is concerned with the reconstruction of d-dimensional objects, such as polyhedral surfaces, from $(d-1)$-dimensional pictures (i.e., projections of the objects onto a hyperplane). Given a $(d-1)$-dimensional picture the task is to assign a dth coordinate to every vertex in the picture such that every face lies in a hyperplane of the d-dimensional space. Such an assignment will always exist as we can choose the last coordinates in such a way that all the vertices lie in the same hyperplane. Two fundamental questions of scene analysis are that can we find an assignment with not all the faces in the same hyperplane or with no two faces in the same hyperplane?

In the generic setting Whiteley [64] answered both of these questions for every $d \geq 2$. In this chapter we investigate the impact of symmetry on the lifting properties of 2-dimensional pictures.

7.1 Basic definitions

We first give the basic definitions of scene analysis.

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$.

A $(d-1)$-*picture* is an incidence structure S together with a corresponding map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, y_i, \ldots, w_i)$, and is denoted by $S(r)$.

A d-*scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \ldots, w_i, z_i)$, and $P : F \to \mathbb{R}^d$, $P^j = (A^j \ldots, C^j, D^j)$, such that for each $(i, j) \in I$ we have $A^j x_i + \ldots + C^j w_i + z_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, \ldots, 0, 1)^T$.)

A *lifting* of a $(d-1)$-picture $S(r)$ is a d-scene $S(p,P)$, with the vertical projection $\Pi(p)=r$. That is, if $p_i=(x_i,\dots,w_i,z_i)$, then $r_i=(x_i,\dots,w_i)=\Pi(p_i)$.

A lifting $S(p,P)$ is *trivial* if all the faces lie in the same plane. Further, $S(p,P)$ is *folded* (or *non-trivial*) if some pair of faces have different planes, and is *sharp* if each pair of faces sharing a vertex have distinct planes. A picture is called *sharp* if it has a sharp lifting. Moreover, a picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

The *lifting matrix* for a picture $S(r)$ is the $|I|\times(|V|+d|F|)$ coefficient matrix $M(S,r)$ of the system of equations for liftings of a picture $S(r)$. For each $(i,j)\in I$ the corresponding row is:

$$
\begin{array}{c|ccc||ccccc|}
 & & v_i & & & & F_j & & \\
(i,j) & 0\dots0 & 1 & 0\dots0 & 0\dots0 & r_i & 1 & 0\dots0 & \\
 & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{|V|}} & \multicolumn{5}{c}{\underbrace{\qquad\qquad\qquad\qquad\qquad}_{d|F|}}
\end{array}
$$

Theorem 7.1.1 *[64, 68] A generic picture of S has independent rows in the lifting matrix if and only if for all non-empty subsets I' of incidences, we have $|I'|\le|V'|+d|F'|-d$.*

Theorem 7.1.2 *[64, 68] A generic picture of an incidence structure $S=(V,F;I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if $|I|=|V|+d|F|-(d+1)$ and $|I'|\le|V'|+d|F'|-(d+1)$ for all subsets I' of incidences with at least two faces.*

Note that it follows from Theorem 7.1.1 that a generic picture of an incidence structure $S=(V,F;I)$ is *minimally flat*, i.e. flat with independent rows in the lifting matrix, if and only if $|I|=|V|+d|F|-d$ and $|I'|\le|V'|+d|F'|-d$ for all non-empty subsets I' of incidences. When S is $(d+1)$-uniform then this condition is equivalent to the $(1,d)$-tightness of S.

As it will be more convenient to talk about hypergraphs instead of incidence structures from now on we will use notation $H=(V,F)$ instead of $S=(V,F;I)$.

In this chapter we will prove the counterpart of Theorem 7.1.1 in the symmetric setting for $d=2$ and point group $\mathcal{C}_3$ for 4-uniform hypergraphs. Let C_3 denote the three-fold rotation. A vertex v of H is said to be *fixed* by C_3 if $C_3v=v$. Similarly, a face $f=\{v_1,\dots,v_m\}$ of H is said to be *fixed* by C_3 if $C_3f=f$, i.e., if $C_3(\{v_1,\dots,v_m\})=\{v_1,\dots,v_m\}$. Finally, an incidence (i,j) of H is said to be *fixed* by C_3 if $C_3((i,j))=(i,j)$. V_3, F_3, I_3 denotes the set of fixed vertices, faces and incidences, respectively. In [49] Schulze showed that if a $\mathcal{C}_3$-symmetric symmetry-generic picture of a 4-uniform hypergraph with $|E|=|V|-3$ is minimally flat then

H is $(1,3)$-tight and $|I_3| = |V_3|$ holds. The main result of this chapter is that these two necessary conditions are sufficient for the existence of a minimally $\mathcal{C}_3$-symmetric flat symmetry-generic picture:

Theorem 7.1.3 *A $\mathcal{C}_3$-symmetric and symmetry-generic picture of a 4-uniform hypergraph H with $|E| = |V| - 3$ has independent rows in the lifting matrix if and only if H is (1,3)-tight and $|V_3| = |I_3|$.*

7.2 A constructive characterization for $\mathcal{C}_3$-symmetric 4-uniform (1,3)-tight hypergraphs

Recall that the *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size $k+1$ incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$. The *j-reduction operation at vertex z on neighbour v* deletes e_1 and replaces e_i by $e_i - z + v$ for all $2 \leq i \leq j+1$.

In this section we show that if H is a 4-uniform $\mathcal{C}_3$-symmetric (1,3)-tight hypergraph with at least seven vertices and with $|I_3| = |V_3|$, then we can always reduce a symmetric set of three vertices such that the resulting smaller hypergraph H' is also $\mathcal{C}_3$-symmetric and (1,3)-sparse also satisfying $|I_3| = |V_3|$. Then assuming that H' has an independent $\mathcal{C}_3$-symmetric realization we also prove that we can find positions for the vertices of H that gives an independent picture.

Suppose that H is a 4-uniform $\mathcal{C}_3$-symmetric hypergraph with the following properties:

$$H \text{ is } (1,3)\text{-tight}; \tag{7.1}$$

$$|I_3| = |V_3|; \tag{7.2}$$

$$|V| > 6. \tag{7.3}$$

Since $|V_3| \leq 1$ 7.2 implies that there are two cases:

1. $|I_3| = |V_3| = 0$;

2. $|I_3| = |V_3| = 1$.

In this section v_0 denotes the fixed vertex and f_0 denotes the fixed hyperedge (note that v_0 and f_0 may not exist).

7.2.1 Preliminaries

We will use one of the key results from Chapter 5:

Theorem 5.2.5 *Let $H = (V, E)$ be a $(1, 3)$-tight 4-uniform hypergraph and let $z \in V$ be a vertex with $d(z) = j$ for some $1 \leq j \leq 3$. Then there is an admissible j-reduction at z.*

We shall also use the following two lemmas:

Lemma 7.2.1 *If H is a $(1, 3)$-sparse 4-uniform hypergraph then H has at least four vertices with degree at most three. Furthermore if there are exactly four vertices with degree at most three then they must have degree one.*

Lemma 7.2.2 *The* def *function is submodular, that is* $\mathrm{def}(X) + \mathrm{def}(Y) \geq \mathrm{def}(X \cup Y) + \mathrm{def}(X \cap Y)$.

The next lemma follows immediately from Lemma 7.2.2.

Lemma 7.2.3 $\mathrm{def}(\mathcal{C}_3 X) \leq 3\mathrm{def}(X) - \mathrm{def}(X \cap C_3 X) - \mathrm{def}(C_3^2 X \cap (X \cup C_3 X))$.

Lemma 7.2.4 *Suppose that $X \subseteq V$ is such that* $\mathrm{def}(X \cap C_3 X) \geq \mathrm{def}(X)$ *and* $\mathrm{def}(Z) \geq \mathrm{def}(X)$ *for any $Z \supseteq X$. Then* $\mathrm{def}(X) = \mathrm{def}(\mathcal{C}_3 X)$.

Proof. By the symmetry of H, Lemma 7.2.2 and the conditions of the lemma we get:

$$2\mathrm{def}(X) = \mathrm{def}(X) + \mathrm{def}(C_3 X) \geq \mathrm{def}(X \cup C_3 X) + \mathrm{def}(X \cap C_3 X) \geq 2\mathrm{def}(X).$$

Which implies that $\mathrm{def}(X \cup C_3 X) = \mathrm{def}(X)$. Furthermore

$$2\mathrm{def}(X) = 2\mathrm{def}(X \cup C_3 X) = \mathrm{def}(X \cup C_3 X) + \mathrm{def}(C_3 X \cup C_3^2 X) \geq$$

$$\geq \mathrm{def}(\mathcal{C}_3 X) + \mathrm{def}((X \cup C_3 X) \cap (C_3 X \cup C_3^2 X)) \geq 2\mathrm{def}(X)$$

from which $\mathrm{def}(X) = \mathrm{def}(\mathcal{C}_3 X)$ follows. □

Lemma 7.2.5 $\mathrm{def}(\mathcal{C}_3 X) \equiv 0, 1 \mod 3$ *for every $X \subseteq V$.*

Proof. By definition $|\mathcal{C}_3 X| - 3 - i(\mathcal{C}_3 X) = \mathrm{def}(\mathcal{C}_3 X)$. The $\mathcal{C}_3$-symmetry implies $|\mathcal{C}_3 X| \equiv 0, 1 \mod 3$ and $i(\mathcal{C}_3 X) \equiv 0, 1 \mod 3$. But $i(\mathcal{C}_3 X) \equiv 1 \mod 3$ implies $|\mathcal{C}_3 X| \equiv 1 \mod 3$ hence $\mathrm{def}(\mathcal{C}_3 X) \equiv 2 \mod 3$ is not possible. □

7.2.2 Reducing low degree vertices

In this section we will define symmetric reductions for $\mathcal{C}_3$-symmetric hypergraphs. We will also prove that there is always a $\mathcal{C}_3$-symmetric reduction that preserves sparsity.

Let $u \in V$ be a vertex not incident with f_0. Suppose that $d(u, v) = 1$ for some $v \in V$. Reduce u on v then reduce C_3u on C_3v and then C_3^2u on C_3^2v. This operation (that consists of three successive reductions) will be called a *$\mathcal{C}_3$-symmetric reduction* or simply a *symmetric reduction* if the group is clear from context. We will say that we *reduce $\mathcal{C}_3u$ on $\mathcal{C}_3v$*. If the resulting hypergraph H' is (1,3)-sparse then the symmetric reduction is called *admissible*.

Lemma 7.2.6 *Let H be a $\mathcal{C}_3$-symmetric 4-uniform hypergraph and $u \in V$ be a vertex not incident with f_0. Hypergraph H' obtained with a $\mathcal{C}_3$-symmetric reductions is $\mathcal{C}_3$-symmetric.*

Proof. It suffices to show that for every hyperedge $f \in E(H')$ we have $C_3f, C_3^2f \in E(H')$. This is clearly true for every hyperedge in $E(H) \cap E(H')$.

If an edge $f_1 \in E(H)$ is incident with both u and v then $f_1, C_3f_1, C_3^2f_1$ are deleted during the reductions. If edge $f_2 \in E(H)$ is incident with u but is nor incident with v then in f_2 u (C_3u and C_3^2u) is replaced with v (C_3v and C_3^2v, respectively)and is not difficult to see that $C_3f_2', C_3^2f_2' \in E(H')$ holds. □

The main result of this section is that we can always find a symmetric set of three vertices for which an admissible symmetric reduction exists. Our first task is to find a vertex u with $d(u) \leq 3$ that is not incident to f_0. By Lemma 7.2.1 the vertices of f_0 are the only vertices with degree at most four, if and only if they all have degree one. But then H has four vertices only. Hence we can always find an appropriate u if $|V| > 6$.

Lemma 7.2.7 *If $d(\mathcal{C}_3u) = 3$ then there is an admissible symmetric reduction at $\mathcal{C}_3u$.*

Proof. The result of an arbitrary symmetric reduction is the deletion of $\mathcal{C}_3u$ together with the incident hyperedges. This reduction is clearly admissible. □

Blockers for symmetric reductions

From now on we will assume that $d(\mathcal{C}_3u) = 6$ or 9 from which $2 \leq d(u) \leq 3$ follows. We will denote the hyperedges incident with u by e_1, e_2 (and e_3 if $d(u) = 3$) and we will also use notation $e_j^- = e_j - u$.

Let $a_1, \ldots, a_l$ denote the neighbors of u in $V - \mathcal{C}_3 u$ for which $d(u, a_i) = 1$, $1 \leq i \leq l$. Note that $d(u) \leq 3$ implies $l \geq 1$. We will also use notation $N_1(u) = \{a_1, \ldots, a_l\}$. The task is to find an index i for which reducing $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ gives a (1,3)-tight hypergraph. The reduced hypergraph H' is not (1,3)-sparse if and only if there is a set of hyperedges $F \subseteq E(H') - E(H)$ for which there is a vertex set $V(F) \subseteq X \subseteq V(H) - \mathcal{C}_3 u$ with $\text{def}(X) \leq |F| - 1$. We will call such a set X a *blocker for the symmetric reduction*. Now we describe the blockers for symmetric reductions. The blocker of a_i will be denoted by X_i.

We will divide blockers into three groups to simplify discussion. Let X_i be a blocker for the symmetric reduction of $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$. We can assume that $a_i \in X_i$ because $\mathcal{C}_3 a_i \cap X_i \neq \emptyset$ and if $a_i \notin X_i$ then we can replace X_i with $C_3 X_i$ or $C_3^2 X_i$ to obtain a blocker that contains a_i.

Vertices u and $C_3 u$ may or may not share a hyperedge. First suppose that there is no hyperedge incident to both u and $C_3 u$. In this case we can not reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ if and only if one of the three following cases occurs.

1. The reduction of u on a_i has a blocker not containing $C_3 u$ and $C_3^2 u$. We will call these *type 1* blockers.

2. Suppose that the reduction of u on a_i has no type 1 blocker and the resulting hypergraph is G_1. If the reduction of $C_3 u$ on $C_3 a_i$ in G_1 has a blocker that does not contain $C_3^2 u$ then such a blocker will be called a *type 2* blocker.

3. If there is no type 1 or type 2 blocker then let the resulting hypergraph be G_2. If the reduction of $C_3^2 u$ on $C_3^2 a_i$ in G_2 has a blocker then that blocker is called a *type 3* blocker.

It follows from the definitions of type 1, 2 and 3 blockers that if X is a type 2 (or type 3) blocker then X must contain the vertex set of at least one (at least two) previously reduced hyperedge. Consider first case $d(u) = 2$. In this case there are three different types of blockers. Let e_t for $1 \leq t \leq 2$ be the edge not incident with a_i. If X_i is type 1 then $\text{def}(X_i) = 0$ and $e_t^- + a_i \subseteq X_i$. If X_i is type 2 then $\text{def}(X_i) = 1$ and $(e_t^- + a_i) \cup C_3(e_t^- + a_i) \subseteq X_i$ while if X_i is type 3 then $\text{def}(X_i) = 2$ and $\mathcal{C}_3(e_t^- + a_i) \subseteq X_i$.

Now suppose that $d(u) = 3$. This implies that $d(C_3 u) = 9$. To simplify notation we will assume that $a_i \in e_1$.

If X_i is type 1 then $\text{def}(X_i) = 0$ or 1. In the former case $e_t^- + a_i \subseteq X_i$ and in the latter case $e_1^- \cup e_2^- + a_i \subseteq X_i$ for some $2 \leq t \leq 3$.

If X_i is type 2 then $1 \leq \text{def}(X_i) \leq 3$. $a_i, C_3 a_i \in X_i$ and X_i contains at least one of sets e_2^- and e_3^- and at least one of $C_3 e_2^-$ and $C_3 e_3^-$ and contains at least $\text{def}(X_i) + 1$

of these four vertex sets. There are two kinds of type 2 blockers that will play an important role in the proofs. The first one is where $\mathrm{def}(X_i) = 1$ and $e_t^- \cup C_3 e_t^- \subseteq X_i$ for some $2 \leq t \leq 3$. We will call such an X_i a *type 2a blocker*. If $\mathrm{def}(X_i) = 1$ and $e_t^- \cup C_3 e_s^- \subseteq X_i$ for $\{s,t\} = \{2,3\}$ then X_i is a *type 2b blocker*.

And finally, if X_i is type 3 then $2 \leq \mathrm{def}(X_i) \leq 5$. $\mathcal{C}_3 a_i \subseteq X_i$ and X_i contains at least one of sets e_2^- and e_3^-, at least one of $C_3 e_2^-$ and $C_3 e_3^-$ and at least one of $C_3^2 e_2^-$ and $C_3^2 e_3^-$. X_i contains at least $\mathrm{def}(X_i) + 1$ of these six vertex sets.

Now suppose that $d(u) = 3$ and u and $C_3 u$ share an edge. $d(u) \leq 3$ implies that u and $C_3 u$ cannot share more than one edge. In this case instead of e_1, e_2, e_3 we will use a different notation for the edges incident to u. Let f be the unique edge incident to both u and $C_3 u$, and so the edges incident to u are $f, C_3 f, g$ for some $g \in F$. We will use notation $f^- = f - u - C_3 u$ and $g^- = g - u$. If $f^- \cap C_3 f^- = \emptyset$ then $(f^- \cup C_3 f^-) \cap N_1(u) \neq \emptyset$ and in this case we will reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ for some $a_i \in f^- \cup C_3 f^-$. If $f^- \cap C_3 f^- \neq \emptyset$ then either $f = \{u, C_3 u, w, C_3 w\}$ or $f = \{u, C_3 u, w, v_0\}$ for some $w \in V - v_0$. In this case $g \cap N_1(u) \neq \emptyset$ and we will reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ for some $a_i \in g^-$.

We will apply the same method as in the case before, that is reducing u on some of its neighbors $a_i \in N_1(u)$ then reducing $C_3 u$ on $C_3 a_i$ and finally $C_3^2 u$ on $C_3^2 a_i$. Note that the first reduction is a 2-reduction but the other ones may be 1-reductions. In either case this sequence of the three operations results in adding exactly three hyperedges to $H - \mathcal{C}_3 u$ in a symmetric way. If $a_i \in f^-$ then let $h_i = g^- + a_i$ and if $a_i \in g^-$ then let $h_i = f^- + a_i + C_3 a_i$. The three new hyperedges are $\mathcal{C}_3 h_i$.

If the reduction is not admissible then again we have three types of blockers. $X_i \subseteq V - \mathcal{C}_3 u$ is a blocker for a_i if one of the following holds:

1. $h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 0$;
2. $h_i \cup C_3 h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 1$;
3. $\mathcal{C}_3 h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 2$.

If $u, C_3 u$ share an edge then we will call these blockers type 1, 2 and 3, respectively.

We shall also use the following property of (1,3)-sparse symmetric hypergraphs throughout this section. Let $U \subseteq V - \mathcal{C}_3 u$ be a vertex set. If $U + u$ spans k edges incident with u then $\mathrm{def}(U) \geq k - 1$ and $\mathrm{def}(\mathcal{C}_3 U) \geq 3(k-1)$ for $2 \leq k \leq 3$. We will call this the $(*)$ property.

From now on we will suppose that a_i has a blocker X_i for every $1 \leq i \leq l$ and X_i will be a blocker with the smallest possible deficiency among blockers of a_i. Note that it follows from the definition of type 1, 2 and 3 blockers that if X_i is type h then there is no type k blocker for a_i with $k < h$.

Case $d(u) = 2$ and $d(\mathcal{C}_3(u)) = 6$

Lemma 7.2.8 *If $d(\mathcal{C}_3 u) = 6$ and $d(u) = 2$ then there is an admissible symmetric reduction at $\mathcal{C}_3 u$.*

Proof. Suppose for a contradiction that there is no symmetric reduction at $\mathcal{C}_3 u$. Then there is a blocker X_i for every a_i.

First we will show that every X_i is type 1 or type 2. Suppose for a contradiction that X_i is type 3. By our assumption $\mathrm{def}(Y) \geq 2$ for every $Y \supseteq \mathcal{C}_3 X_i$ and $\mathrm{def}(X_i \cap C_3 X_i) \geq 2$ and hence we can use Lemma 7.2.4. We get that $\mathrm{def}(\mathcal{C}_3 X_i) = 2$ which contradicts Lemma 7.2.5 hence X_i is type 1 or type 2 as we claimed.

Now suppose that X_i is type 2 for some $1 \leq i \leq l$. If $X_i \cap C_3 X_i$ is tight then a_i has a type 1 blocker which is not possible, hence we must have $\mathrm{def}(X_i \cap C_3 X_i) \geq 1$. We can use again Lemma 7.2.4 to obtain $\mathrm{def}(\mathcal{C}_3 X_i) = 1$.

By Theorem 5.2.5 it is not possible that every blocker is type 1. Therefore we can assume that X_1 is type 2. Assume further that $a_1 \in e_1$. Suppose first that X_j is type 1 for every $a_j \in e_2$. Then $\bigcup_{a_j \in e_2} X_j$ is a tight set by Lemma 7.2.2 and contains every neighbor of u which contradicts the $(*)$ property. Hence there must be an $a_2 \in e_2$ for which X_2 is type 2. Consider sets $\mathcal{C}_3 X_1$ and $\mathcal{C}_3 X_2$. $\mathrm{def}(\mathcal{C}_3 X_1) = \mathrm{def}(\mathcal{C}_3 X_2) = 1$ and $|\mathcal{C}_3 X_1 \cap \mathcal{C}_3 X_2| \geq 4$. This implies $\mathrm{def}(\mathcal{C}_3 X_1 \cup \mathcal{C}_3 X_2) \leq 2$ by Lemma 7.2.2 and so $\mathcal{C}_3 X_1 \cup \mathcal{C}_3 X_2$ violates sparsity by the $(*)$ property. This completes the proof. $\square$

Case $d(u) = 3$ and $d(\mathcal{C}_3(u)) = 9$

Claim 7.2.9 *Suppose X_i is type 3. Then $\mathrm{def}(X_i) \leq 4$ and if $\mathcal{C}_3 e_t^- \subseteq X_i$ for some $1 \leq t \leq 3$ then $\mathrm{def}(X_i) \geq 3$.*

Proof. Suppose that there is a type 3 blocker X_i with $\mathrm{def}(X_i) = 5$ or with $\mathcal{C}_3 e_t^- \subseteq X_i$ for some $1 \leq t \leq 3$ and $\mathrm{def}(X_i) = 2$. In both of these cases we can use Lemma 7.2.4 to deduce $\mathrm{def}(\mathcal{C}_3 X_i) = \mathrm{def}(X_i)$ which contradicts Lemma 7.2.5. $\square$

The next claim follows easily from Lemma 7.2.4.

Claim 7.2.10 *If X_i is a type 2a blocker then $\mathrm{def}(\mathcal{C}_3 X_i) = 1$.*

Lemma 7.2.11 *Suppose that $Y \subseteq V$ is such that $\mathrm{def}(\mathcal{C}_3 Y) \leq 4$ and $\mathcal{C}_3(e_t^- \cup e_s^-) \subseteq Y$ for some pair $1 \leq t, s \leq 3$. If $a_i \notin e_t^- \cup e_s^-$ then $\mathrm{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_i) \leq 4$.*

Proof. It suffices to show that $\mathrm{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_i) \leq 5$ because the statement then follows from Lemma 7.2.5.

If X_i is type 1 and tight, then we can use Lemma 7.2.2 three times to deduce $\mathrm{def}(\mathcal{C}_3Y \cup \mathcal{C}_3X_i) \leq \mathrm{def}(\mathcal{C}_3Y)$. If X_i is type 1 with $\mathrm{def}(X_i) = 1$ then $\mathrm{def}(\mathcal{C}_3Y \cap X_i) \geq 1$ must hold by the $(*)$ property. Similarly to the previous case using Lemma 7.2.2 three times we can get $\mathrm{def}(\mathcal{C}_3Y \cup \mathcal{C}_3X_i) \leq \mathrm{def}(\mathcal{C}_3Y)$.

If X_i is type 2a then $\mathrm{def}(\mathcal{C}_3X_i) = 1$ and hence $\mathrm{def}(\mathcal{C}_3Y \cup \mathcal{C}_3X_i) \leq \mathrm{def}(\mathcal{C}_3Y) + 1 \leq 5$ by Lemma 7.2.2. If X_i is type 2b then $\mathrm{def}(\mathcal{C}_3Y \cup X_i) \leq \mathrm{def}(\mathcal{C}_3Y) + 1$. We also have $\mathrm{def}((\mathcal{C}_3Y \cup X_i) \cap C_3X_i) \geq 1$ and $\mathrm{def}((\mathcal{C}_3Y \cup X_i \cup C_3X_i) \cap C_3^2X_i) \geq 1$. These imply $\mathrm{def}(\mathcal{C}_3Y \cup \mathcal{C}_3X_i) \leq \mathrm{def}(\mathcal{C}_3Y \cup X_i \cup C_3X_i) \leq \mathrm{def}(\mathcal{C}_3Y \cup X_i) \leq \mathrm{def}(\mathcal{C}_3Y) + 1$.

If X_i is not type 1, type 2a or type 2b then $\mathrm{def}(X_i) \geq 2$ holds.

Claim 7.2.12 *If* $\mathrm{def}(X_i) \geq 2$ *then* $\mathrm{def}(\mathcal{C}_3X_i) \leq 4$ *holds.*

Proof. Again, it suffices to show that $\mathrm{def}(\mathcal{C}_3X_i) \leq 5$. We split the proof into several cases. In each case we will use Lemma 7.2.3 and the fact that X_i is a blocker with the smallest deficiency.

If X_i is type 2 and $\mathrm{def}(X_i) = 2$ then $\mathrm{def}(\mathcal{C}_3X_i) \leq 6 - 1 - 1$. If $\mathrm{def}(X_i) = 3$ then $\mathrm{def}(\mathcal{C}_3X_i) \leq 9 - 2 - 2$.

Now suppose that X_i is type 3. Suppose first that $\mathrm{def}(X_i) = 2$. By the symmetry and Claim 7.2.9 we can assume that X_i contains the vertices of $e_t^-, C_3e_t^-, C_3^2e_s^-$. Then $\mathrm{def}(\mathcal{C}_3X_i) \leq 6 - 1 - 1$.

If $\mathrm{def}(X_i) = 3$ then there are two cases. We can assume in the first case that X_i contains $\mathcal{C}_3e_t^-, e_s^-$ while in the second one it contains $e_t^-, e_s^-, C_3e_s^-, C_3^2e_t^-$. In the first case we have $\mathrm{def}(\mathcal{C}_3X_i) \leq 9 - 3 - 3$ and in the second one $\mathrm{def}(\mathcal{C}_3X_i) \leq 9 - 2 - 2$.

And finally if $\mathrm{def}(X_i) = 4$ then $\mathrm{def}(\mathcal{C}_3X_i) \leq 12 - 4 - 4$. $\mathrm{def}(X_i) = 5$ is not possible by Claim 7.2.9 and this completes the proof. □

If $\mathrm{def}(X_i) \geq 2$ then $\mathrm{def}(\mathcal{C}_3X_i) \leq 4$ by Claim 7.2.12. In this case $\mathrm{def}(\mathcal{C}_3Y \cap \mathcal{C}_3X_j) \geq 3$ follows from the $(*)$ property. Then by Lemma 7.2.3 $\mathrm{def}(\mathcal{C}_3Y \cup \mathcal{C}_3X_j) \leq 4 + 4 - 3$ and this completes the proof. □

Lemma 7.2.13 *Suppose there is a set* $Y \subseteq V$ *with* $\mathrm{def}(\mathcal{C}_3Y) \leq 4$ *and* $\mathcal{C}_3(e_t^- \cup e_s^-) \subseteq Y$ *for some pair* $1 \leq t, s \leq 3$. *Then* H *is not (1,3)-sparse.*

Proof. Using Lemma 7.2.11 we get that $\mathrm{def}(\mathcal{C}_3Y \bigcup_{j:a_j \in N_1(u) \setminus \mathcal{C}_3Y} \mathcal{C}_3X_j) \leq 4$ and hence set $\mathcal{C}_3Y \bigcup_{j:a_j \in N_1(u) \setminus \mathcal{C}_3Y} \mathcal{C}_3X_j$ violates sparsity by the $(*)$ property. □

Lemma 7.2.14 $\mathrm{def}(X_i) \leq 1$ *holds for every blocker* X_i *and if* $\mathrm{def}(X_i) = 1$ *then* X_i *is type 2.*

Proof. Suppose for a contradiction that $2 \leq \text{def}(X_i)$ for some $1 \leq i \leq l$. Then $\text{def}(\mathcal{C}_3 X_i) \leq 4$ by Claim 7.2.12. Using Claim 7.2.9 we get that $\mathcal{C}_3 X_i$ contains $\mathcal{C}_3 e_s^-$ and $\mathcal{C}_3 e_t^-$ for some pair $1 \leq s, t \leq 3$. Then using Lemma 7.2.13 we get a contradiction.

For the second part of the statement suppose that X_i is type 1 with $\text{def}(X_i) = 1$. Suppose $a_i \in e_1^-$. If X_j is type 1 for every $a_j \in e_1^-$, $i \neq j$ then it can be seen easily using Lemma 7.2.2 and the $(*)$ property that $\text{def}(\bigcup_{j:a_j \in e_1^-} X_j) \leq 1$ which is a contradiction. Hence there is some $a_k \in e_1^-$ with a type 2 blocker X_k. If X_k is type 2a then $\text{def}(\mathcal{C}_3 X_k) = 1$ by Claim 7.2.10 from which $\text{def}(\mathcal{C}_k(X_j \cup X_k)) \leq 4$ follows which contradicts Lemma 7.2.13. While if X_k is type 2b then consider set $X_j \cup X_k$. $\text{def}(X_j \cup X_k) \leq 2$ and $\text{def}((X_j \cup X_k) \cap \mathcal{C}_3(X_j \cup X_k)) \geq 0$ and $\text{def}((X_j \cup X_k) \cup \mathcal{C}_3(X_j \cup X_k) \cap \mathcal{C}_3^2(X_j \cup X_k)) \geq 1$. Hence using Lemma 7.2.3 and 7.2.5 $\text{def}(\mathcal{C}_3(X_j \cup X_k)) \leq 4$ and again we get a contradiction using Lemma 7.2.13 which completes the proof. □

We have shown so far that every blocker has to be a tight type 1 blocker, a type 2a or a type 2b blocker. We shall also use the following lemma.

Lemma 7.2.15 *Suppose that $|e_j^- \cap e_k^-| \geq 1$ and $e_k^- \subseteq X_i$ for some $a_i \in e_j^-$. Then X_i is not type 2a.*

Proof. Suppose for a contradiction that X_i is type 2a. Then $\text{def}(\mathcal{C}_3 X_i) = 1$ by Lemma 7.2.4. Thus $\text{def}(\mathcal{C}_3(X_i \cup e_j^-)) \leq 4$ which is a contradiction by Lemma 7.2.13. □

Now we will show that if there is a blocker X_i of a_i for every $1 \leq i \leq l$ then H cannot be $(1,3)$-sparse.

Lemma 7.2.16 *If $d(u) = 3$, u and $C_3 u$ do not share a hyperedge then there is an admissible symmetric reduction at $\mathcal{C}_3 u$.*

Proof. Suppose for a contradiction that there is no admissible symmetric reduction at $\mathcal{C}_3 u$. Then a_i has a blocker for every $1 \leq i \leq l$. By Lemma 7.2.14 X_i is a tight type 1 or type 2a or type 2b blocker for every $1 \leq i \leq l$. By Theorem 5.2.5 there is a (non-symmetric) admissible reduction at u hence we can assume that a_1 has no type 1 blocker and hence X_1 is a type 2 blocker.

Case 1: Suppose first that every blocker is either type 1 or type 2a.

Claim 7.2.17 *Suppose that X_j is type 1 or type 2a for every $a_j \in e_s^-$ and $e_s^- \cup e_t^- \subseteq \bigcup_{a_j \in e_s^-} X_j$. Then $e_t^- \not\subseteq \bigcap_{a_j \in e_s^-} X_j$.*

Proof. Suppose the contrary for a contradiction. If X_i is type 2a for some $a_i \in e_s^-$ then $\text{def}(\mathcal{C}_3 X_i) \leq 1$ by Claim 7.2.10 and we can easily deduce that $\text{def}(\bigcup_{a_j \in e_s^-} \mathcal{C}_3 X_j) \leq 3$ using Lemma 7.2.2 and then we get a contradiction using Lemma 7.2.11. If X_j is type 1 for every j then we have $\text{def}(\bigcup_{a_j \in e_s^-} X_j) \leq 1$ which contradicts sparsity by the $(*)$ property. $\square$

We first claim that $|e_j^- \cap N_1(u)| \leq 2$ for every $1 \leq j \leq 3$. Suppose for a contradiction that $e_1^- \cap (e_2^- \cup e_3^-) = \emptyset$. Then $e_1^- = \{a_i, a_j, a_k\}$ for some triple $1 \leq i, j, k \leq l$. It follows from Claim 7.2.17 that $X_i \cap X_j \cap X_k \supseteq e_t^-$ is not possible if $t \in \{2, 3\}$. Hence we can assume that $X_i \not\supseteq e_2^-$ and $X_j \not\supseteq e_3^-$. Assume $X_k \supseteq e_2^-$. $e_1^- \cap (e_2^- \cup e_3^-) = \emptyset$ implies $|e_2^- \cap N_1(u)| \geq 1$ hence $a_m \in e_2^-$ for some $1 \leq m \leq l$. If $X_m \supseteq e_1^-$ then we claim that $X_j \cup X_k \cup X_m$ violates sparsity. If X_j, X_k, X_m are type 1 blockers then $X_j \cup X_k \cup X_m$ is tight and hence violates sparsity. If at least one of X_j, X_k, X_m is type 2a then $\text{def}(\mathcal{C}_3(X_j \cup X_k \cup X_m)) \leq 3$ by Lemma 7.2.2 and the $(*)$ property which violates sparsity by Lemma 7.2.11.

Hence $X_m \supseteq e_3^-$ for every $a_m \in e_2^-$ but this contradicts Claim 7.2.17. We deduce that $|e_j^- \cap N_1(u)| \leq 2$ for every $1 \leq j \leq 3$ which implies $|N(u)| \leq 7$. There is a type 2a blocker hence $e_j^- \cap e_k^- = \emptyset$ for some pair $1 \leq j, k \leq 3$ by Lemma 7.2.15. This implies $|N(u)| \geq 6$.

If $|N(u)| = 6$ then $l \geq 3$. Consider sets X_1, X_2. X_1 is type 2a by our assumption, hence $\text{def}(\mathcal{C}_3 X_1) = 1$ by Claim 7.2.10. If $N(u) \subseteq X_1 \cup X_2$ then $|X_1 \cap X_2| \geq 2$. Hence if X_2 is type 1 then $\text{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 4$ and if X_2 is type 2a then $\text{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 2$. In both cases we get a contradiction. If $|N(u) \cap (X_1 \cup X_2)| = 5$ then $|X_1 \cap X_2| \geq 3$. If X_2 is type 1 then $\text{def}(\mathcal{C}_3(X_1 \cup X_2)) = 1$ and hence $\text{def}(\mathcal{C}_3(N(u) \cup X_1 \cup X_2)) \leq 4$ while if X_2 is type 2a then $\text{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 2$ and hence $\text{def}(\mathcal{C}_3(N(u) \cup X_1 \cup X_2)) \leq 5$. These contradict sparsity by the $(*)$ property.

The remaining case is where $|N(u)| = 7$ and $|e_j^- \cap N_1(u)| \leq 2$ for every $1 \leq j \leq 3$. The only possible configuration is where $e_1^- \cap e_2^- \cap e_3^- = \emptyset$ and $|e_1^- \cap e_2^-| = |e_2^- \cap e_3^-| = 1$, thus $|e_2^- \cap N_1(u)| = 1$ and $|e_1^- \cap N_1(u)| = |e_3^- \cap N_1(u)| = 2$. If $a_1 \in e_2^-$ then we get a contradiction by Lemma 7.2.15. Hence we can assume that $a_1 \in e_1^-$ and $e_3^- \subseteq X_1$. By Claim 7.2.17 there is an $a_j \in e_3^-$ with $e_1^- \subseteq X_j$. If X_j is type 1 then $\text{def}(\mathcal{C}_3(X_1 \cup X_j)) \leq 4$ and if X_j is type 2a then $\text{def}(\mathcal{C}_3(X_1 \cup X_j)) \leq 2$. Both lead to a contradiction by Lemma 7.2.13.

Case 2: It remains to consider the case where X_1 is a type 2b blocker. We can assume that $e_3^- \cup C_3 e_2^- \subseteq X_1$ and $a_1 \in e_1^-$.

Claim 7.2.18 *If X_i is type 2b then $X_i \cap C_3 X_i = 1$.*

Proof. Suppose that $X_i \cap C_3 X_i \geq 2$ for a type 2b blocker X_i. Then by Lemma 7.2.3

$\text{def}(\mathcal{C}_3 X_i) \leq 3+1+1$ and hence $\text{def}(\mathcal{C}_3 X_i) \leq 4$ by Lemma 7.2.5. Then we get a contradiction using Lemma 7.2.11. $\square$

$e_2^- \cap e_3^- = \emptyset$ by Claim 7.2.18. We first claim that there is a vertex $a_2 \in e_2^-$ for which X_2 is type 2b. Suppose that X_j is type 1 or type 2a for every vertex in $e_2^- \cap N_1(u)$. Then by Claim 7.2.17 there must be an $a_k \in e_2^-$ for which $e_3^- \subseteq X_k$. But if X_k is type 1 then $X_1 \cup X_k$ is a type 2b blocker for a_1 which contradicts Claim 7.2.18 and if X_k is type 2a then $\text{def}(\mathcal{C}_3(X_1 \cup X_k)) \leq 4$ which contradicts Lemma 7.2.13. Hence there is a vertex, say $a_2 \in e_2^-$ for which X_2 is type 2b. Using a similar argument we can conclude that there is an $a_3 \in e_3^-$ with a type 2b blocker X_3.

Then by Claim 7.2.18 sets e_1^-, e_2^-, e_3^- are pairwise disjoint hence $|N_1(u)| = 9$. It also follows from the argument above that every blocker must be type 2b.

Now suppose that $e_2^- \cup C_3 e_3^- \subseteq X_4$ for some $a_4 \in e_1^-$. Then $\text{def}(X_1 \cup C_3 X_4) \leq 2$ and $\text{def}((X_1 \cup C_3 X_4) \cap C_3(X_1 \cup C_3 X_4)) \geq 1$. This implies $\text{def}(\mathcal{C}_3(X_1 \cup X_4)) \leq 4$ by Lemma 7.2.3 which contradicts Lemma 7.2.13. Hence $e_3^- \cup C_3 e_2^- \subseteq X_1 \cap X_4 \cap X_7$ with notation $e_1^- = \{a_1, a_4, a_7\}$.

We can use a similar argument as above if $e_3^- \cup C_3 e_1^- \subseteq X_2$ for $a_2 \in e_2^-$ to deduce $\text{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 4$ and get a contradiction. Hence $e_1^- \cup C_3 e_3^- \subseteq X_2$ is the only possible case. Then consider $\mathcal{C}_3 X_2 \cup X_1 \cup C_3 X_4 \cup C_3^2 X_7$ which contains $\mathcal{C}_3 N(u)$. We will prove that this set violates sparsity. $\text{def}(X_2 \cup C_3 X_4) \leq 2$ and adding sets $C_3 X_2, C_3^2 X_7, C_3^2 X_2$ in this order we can easily conclude $\text{def}(X_2 \cup C_3 X_4 \cup C_3 X_2 \cup C_3^2 X_7 \cup C_3^2 X_2) \leq 5$ because each set intersects the union of the previous ones in at least three vertices. $e_3^- + a_1 \subseteq (X_2 \cup C_3 X_4 \cup C_3 X_2 \cup C_3^2 X_7 \cup C_3^2 X_2) \cap X_1$ hence $\text{def}(\mathcal{C}_3 X_2 \cup X_1 \cup C_3 X_4 \cup C_3^2 X_7) \leq 5$. In each case we got a contradiction hence we can always perform a symmetric reduction as we claimed. $\square$

Case $d(u) = 3$ and $d(\mathcal{C}_3(u)) = 6$

Lemma 7.2.19 *Suppose that $d(u) = 3$ and the hyperedges incident to u are $f, C_3 f, g$. Then there is a vertex $a_i \in N_1(u)$ such the reduction of $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ is admissible. Moreover, a_i can be chosen such that if $f^- \cap C_3 f^- = \emptyset$ then $a_i \in f^- \cup C_3 f^-$ and if $f^- \cap C_3 f^- \neq \emptyset$ then $a_i \in g^-$.*

Proof. Suppose for a contradiction that there is a blocker X_i for every $1 \leq i \leq l$. It follows easily from Lemmas 7.2.5 and 7.2.4 every blocker is type 1 or type 2. Suppose first that $f^- \cap C_3 f^- = \emptyset$. Consider the blockers for $a_i \in f^- \cup C_3 f^-$. Let $Y = \bigcup_{i: a_i \in f^- \cup C_3 f^-} X_i$. $N(u) \subseteq Y$ hence $\text{def}(Y) \geq 2$ and $\text{def}(\mathcal{C}_3 Y) \geq 6$ must hold by the $(*)$ property. But if X_i is type 1 for every $a_i \in f^- \cup C_3 f^-$ then Y is tight.

If there is some $a_i \in f^- \cup \mathcal{C}_3 f^-$ for which X_i is type 2 then $\mathrm{def}(\mathcal{C}_3 Y) \leq 3$ follows easily so in both cases we get a contradiction.

Now suppose that $f^- \cap \mathcal{C}_3 f^- \neq \emptyset$. Consider blockers X_i for every $a_i \in g^-$. In this case by using Lemma 7.2.3 we can deduce $\mathrm{def}(\mathcal{C}_3 X_i) \leq 1$ for every $a_i \in g^-$. Hence $\mathrm{def}(\bigcup_{i: a_i \in g^-} X_i) \leq 3$ follows which is a contradiction. This completes the proof. $\square$

If we combine the results of Lemmas 7.2.7, 7.2.8, 7.2.16 and 7.2.19 we get the following:

Theorem 7.2.20 *Let $H = (V, F)$ be a $\mathcal{C}_3$-symmetric (1,3)-tight hypergraph with $|V_3| = |I_3|$ and let $u \in V$ be a vertex with $d(u) \leq 3$ not incident to f_0. Then there is a $\mathcal{C}_3$-symmetric admissible reduction at $\mathcal{C}_3 u$.*

7.3 Preserving independence in the lifting matrix

In this section we first show that if H is a 4-uniform hypergraph with an independent 2-picture and H' is obtained from H by a j-extension for some $j \geq 0$ then H' also has an independent 2-picture.

Theorem 7.3.1 *Let (H, r) be an independent 2-picture where $H = (V, F)$ is a 4-uniform hypergraph and $r : V \to \mathbb{R}^2$ is a location map. Let $H' = (V', F')$ be the hypergraph obtained from H by performing a j-extension at $v \in V$ such that $V' = V + z$ and $\{a, b, v, z\} \in F'$. Put $r(z) = r(v)$. If $r(a), r(b), r(v)$ do not lie on a line then (H', r) is an independent 2-picture.*

Proof. (H, r) is an independent 2-picture if and only if the rows of $M(H, r)$ are independent. We have to show that the rows of $M(H', r)$ are also independent.

$$\begin{array}{c|ccccc|cccccc|}
 & \mathrm{z} & \mathrm{v} & \mathrm{a} & \mathrm{b} & & & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{e'_i}} & & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{e}} \\
\cline{2-12}
 & & & & \star & & & & \star & & & \\
(z, e'_i) & 1 & 0 & 0 & 0 & 0\ldots0 & 0\ldots0 & r(z) & 1 & 0\ldots0 & \underline{0} & 0 \\
 & & & & \star & & & & \star & & & \\
\cline{2-12}
(z, e) & 1 & 0 & 0 & 0 & & & & & & r(z) & 1 \\
(v, e) & 0 & 1 & 0 & 0 & 0 & & & 0 & & r(v) & 1 \\
(a, e) & 0 & 0 & 1 & 0 & & & & & & r(a) & 1 \\
(b, e) & 0 & 0 & 0 & 1 & & & & & & r(b) & 1 \\
\cline{2-12}
\end{array} = M(H', r)$$

$M(H', r)$ can be constructed from $M(H, r)$ as follows. First add 4 zero columns, one of which corresponds to z and the rest of them correspond to e. Clearly,

this operation results in a row-independent matrix. Then add the rows of incidences $(v, e), (a, e), (b, e)$. The rows of the matrix obtained are independent since $r(a), r(b), r(v)$ do not lie on a line. Then adding the row of (z, e) preserves the independence because no other row has a non-zero element in the first column.

Now observe that what is left is to modify the rows corresponding to incidences (z, e'_i) for $0 \leq i \leq j$. We can obtain the desired row of (z, e'_i) by subtracting the row of (v, e) and adding the row of (z, e). These operations also preserve independence and this completes the proof. $\square$

7.3.1 Base graphs and $\mathcal{C}_3$-symmetric extensions

First we shall see that the lifting matrices corresponding to the base hypergraphs have full rank. Observe that for a hypergraph H with one edge only the first four columns of $M(H, r)$ form an identity matrix and hence its rows are independent. Thus, in the first case where the base hypergraph H has only one hyperedge and $|V_3| = |I_3| = 1$ we are ready.

In the second case H has six vertices and three hyperedges and $|V_3| = |I_3| = 0$. Hence we have two vertex orbits, $\mathcal{C}_3 v_1$ and $\mathcal{C}_3 v_2$. There are two possible configurations with these properties. In the first one $F = \mathcal{C}_3(\mathcal{C}_3 v_1 + v_2)$ and in the second one $F = \mathcal{C}_3(v_1, v_2, C_3 v_1, C_3 v_2)$. For both of these configurations we will construct a row-independent $\mathcal{C}_3$-symmetric realization using Theorem 7.3.1. Let $r(v_1) \neq (0, 0)$ be arbitrary, place $\mathcal{C}_3 v_1$ symmetrically.

In the first case start with hyperedge $\{\mathcal{C}_3 v_i, v_2\}$ and $r(v_1) = r(v_2)$. Then add $\{\mathcal{C}_3 v_i, C_3 v_2\}$ with $r(C_3 v_1) = r(C_3 v_2)$ and finally add $\{\mathcal{C}_3 v_i, C_3^2 v_2\}$ with $r(C_3^2 v_1) = r(C_3^2 v_2)$. This realization is row-independent by Theorem 7.3.1.

In the second case we put $r(v_2) = r(C_3 v_1)$ (and then $r(C_3 v_2) = r(C_3^2 v_1)$, $r(C_3^2 v_2) = r(v_1)$). We start again with hyperedge $\{\mathcal{C}_3 v_i, C_3 v_2\}$. Then apply a 1-extension at $C_3^2 v_1$. This results in deleting the only edge and adding $\{v_1, v_2, C_3 v_1, C_3 v_2\}$ and $\{\mathcal{C}_3 v_i, C_3 v_2\}$. After one more 1-extension at v_1 we obtain the hypergraph with the desired edges. Both of these extensions satisfy the conditions of Theorem 7.3.1 hence we can conclude that this realization is row-independent.

We shall prove that inverse operations of symmetric reductions defined in Section 7.2.2 preserve the row-independence of the lifting matrix. These inverse operations will be called *symmetric extensions*. Let us recall that the *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size 4 incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$.

Lemma 7.3.2 *Every $\mathcal{C}_3$-symmetric extension preserves the independence of the rows of the lifting matrix.*

Proof. If we apply three j-extensions on H such that the 3 new vertices do not share an edge, then we can apply Theorem 7.3.1 three times to see that the resulting symmetric hypergraph has an independent symmetric 2-picture.

In the second case we can use a similar argument since the new hyperedge always satisfies the conditions of Theorem 7.3.1. □

If we combine Theores 7.2.20 and Lemma 7.3.2 then we get 7.1.3.

Chapter 8

Summary

The dissertation focuses on combinatorial problems connected to combinatorial rigidity. In the first part of this work we consider bar-and-joint frameworks. A bar-and-joint framework is *flexible* if there is a continuous motion that changes the distance between at least one pair of joints without changing the lengths of the bars. Otherwise the framework is *rigid*.

We show that if $\mathcal{R}_2(G)$ (the two-dimensional rigidity matroid of graph G) is 11-connected then $\mathcal{R}_2(G)$ uniquely determines G. This is a corollary of the following statement considering graph connectivity: if G is 7-connected then it is uniquely determined by $\mathcal{R}_2(G)$. We provide a sharp upper bound for the number of edges of minimally k-rigid graphs in $\mathbb{R}^d$ for all k. (These are graphs that remain rigid in $\mathbb{R}^d$ after the deletion of at most $k-1$ arbitrary vertices.) We also give lower bounds for arbitrary values of k and d and show its sharpness for the cases where $k = 2$ and d is arbitrary and where $k = d = 3$. Next we focus on the rigidity of two different types of non-generic planar frameworks. We give a characterization for the existence of an infinitesimally rigid two-dimensional realization with two designated vertices coincident. We characterize symmetry-forced rigid graphs in the plane for every cyclic point group and for dihedral point groups $\mathcal{D}_k$ if k is odd. Both of these results are based on an inductive construction.

In the second part of the thesis we investigate the rigidity of frameworks on hypergraphs. We develop a new inductive construction of 4-regular (1,3)-tight hypergraphs. Using this construction we characterize projectively rigid hypergraphs on the projective line. We also prove a $\mathcal{C}_3$-symmetric version of this hypergraph construction and use it to characterize minimally flat $\mathcal{C}_3$-symmetric 2-pictures - pictures that are not the projection of a non-trivial three-dimensional polyhedral surface.

Chapter 9

Összefoglaló (Summary in Hungarian)

A dolgozat témája gráfok és szerkezetek merevségével kapcsolatos kérdések vizsgálata. A fejezetek többségében rúd-csukló szerkezetekkel foglalkozunk. Egy rúd-csukló szerkezetet *merevnek* mondunk, ha nincsen folytonos deformációja, azaz olyan mozgása, amely megőrzi a rudak hosszát, de megváltoztatja legalább két csukló távolságát.

Bebizonyítjuk, hogy ha $\mathcal{R}_2(G)$ (a G gráf kétdimenziós merevségi matroidja) 11-összefüggő, akkor $\mathcal{R}_2(G)$ egyértelműen meghatározza G-t. Ez az alábbi eredményből következik: ha G 7-összefüggő, akkor $\mathcal{R}_2(G)$ meghatározza G-t. Adunk egy éles felső korlátot az $\mathbb{R}^d$-ben minimálisan k-merev gráfok élszámára minden d-re és k-ra. (Akkor nevezünk egy gráfot k-merevnek $\mathbb{R}^d$-ben, ha legfeljebb $k-1$ tetszőleges pontjának törlése után merev marad $\mathbb{R}^d$-ben.) Alsó korlátot is adunk minden k-ra és d-re és megmutatjuk, hogy ezek a korlátok élesek $k = 2$-re tetszőleges d mellett, valamint $k = d = 3$ esetén.

Két különböző nem-generikus síkbeli szerkezet merevségével is foglalkozunk. Karakterizáljuk azon gráfokat, amelyeknek van olyan merev realizációja a síkban, ahol két kijelölt pont helyzete azonos. Szintén karakterizáljuk azon gráfokat, melyek szimmetrikusan merevek a síkban a forgatáscsoportok és azon $\mathcal{D}_k$ diédercsoportok esetén, amelyekre k páratlan. Mindkét eredmény alapja egy-egy konstruktív karakterizáció.

A dolgozat hipergráfokon értelmezett szerkezetekkel is foglalkozik. Először a 4-reguláris (1,3)-kritikus hipergráfok osztályára adunk egy konstruktív karakterizációt. Ezt az eredményt használva jellemezzük a projektív egyenesen 'projektív merev' hipergráfokat. A hipergráf-konstrukció egy $\mathcal{C}_3$-szimmetrikus változatát is bebizonyítjuk, hogy jellemezni tudjuk a minimálisan nem-felemelhető síkbeli $\mathcal{C}_3$-szimmetrikus kép-szerkezeteket.

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