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On the Stokes problem

PhD Thesis

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Contents

1 Introduction

The Navier-Stokes equations are the fundamental equations of motion of a viscous fluid. These equations describe how the velocity, pressure and density of a moving fluid are related. The derivation of these equations is based on Newton's generalized law of friction, according to which the stress in a moving fluid or gas is proportional to the strain rates (Newtonian fluid), the constant of this proportionality being the viscosity.

For a flow of an incompressible liquid (i.e. the density is constant) these equations can be put in the following vector form:

$$\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial \vec{u}}{\partial x_j} + \operatorname{grad} p = \nu \Delta \vec{u} + \vec{f}, \tag{1}$$

$$\operatorname{div} \vec{u} = 0, \tag{2}$$

where n is the dimension of the domain, $\vec{u}(t,\vec{x}) = (u_1(t,\vec{x}), \ldots u_n(t,\vec{x}))^T$ is the velocity vector function, $p = p(t,\vec{x})$ is the kinematic pressure function (the pressure divided by the density ρ), $\rho\vec{f}$ describes the density of the body forces and ν denotes the kinematic viscosity (the viscosity divided by the density). They are mathematical expressions of the conservation laws of momentum (1) and mass (2). In practice it is worth to transform these equations into a dimensionless form, scaling all quantities figuring therein by dividing by appropriate characteristic units. After that, the Navier-Stokes equations contain one decisive dimensionless parameter known as the Reynolds number

$$\mathrm{Re} = \frac{u_0 \ell_0}{\nu},$$

where u_0 is a characteristic velocity and ℓ_0 is a characteristic length. In order to specify more closely a problem, one must also prescribe initial and boundary conditions for the unknown velocities, and, e.g., the mean value of the pressure.

The Navier-Stokes equations are nonlinear partial differential equations which nonlinearity is due to convective acceleration $\sum_{j=1}^{n} u_j \frac{\partial \vec{u}}{\partial x_j}$ in (1), which is an acceleration associated to the change of velocity over position. This nonlinearity makes it difficult or impossible to solve them analytically. Neglecting nonlinearity, represents a considerable simplification of the full Navier-Stokes equations. Moreover, if one assumes the inertial forces (i.e. $\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial \vec{u}}{\partial x_j}$) to be negligible then one achieves the stationary Stokes equations (let $\nu = 1$)

$$-\Delta \vec{u} + \operatorname{grad} p = \vec{f},$$

$$\operatorname{div} \vec{u} = 0,$$

which can be solved for the unknown velocity $\vec{u}$ and the corresponding pressure p in appropriate chosen function spaces if one prescribes boundary values for the velocity along the boundary of the planar or spatial domain, where the equations must be solved. The Stokes problem is called of first kind if one prescribes Dirichlet boundary conditions for the velocity. Note further that the pressure p can be determined uniquely only up to an additive constant. Therefore one imposes mostly a normalization constraint: one can prescribe the value of the pressure in a point of the domain or one can demand the integral of the pressure over the domain to be zero.

The first kind Stokes problem is much simpler to solve as the Navier-Stokes problem, because it is linear allowing superposition of solutions. This means, if $\vec{u}_1$ and $\vec{u}_2$ solve in the domain Ω with boundary denoted by $\partial\Omega$ and with corresponding pressures p_1 and p_2 the first kind Stokes problems

$$\begin{aligned} -\Delta\vec{u}_1 + \operatorname{grad} p_1 &= \vec{f}, \text{ and } \operatorname{div}\vec{u}_1 = 0 \text{ in } \Omega, \\ \vec{u}_1 &= 0, \text{ on } \partial\Omega \end{aligned}$$

(i.e. inhomogeneous momentum equation and homogeneous boundary condition) and

$$\begin{aligned} -\Delta\vec{u}_2 + \operatorname{grad} p_2 &= 0, \text{ and } \operatorname{div}\vec{u}_2 = 0 \text{ in } \Omega, \\ \vec{u}_2 &= \vec{u}_0, \text{ on } \partial\Omega \end{aligned}$$

(i.e. homogeneous momentum equation and inhomogeneous boundary condition with prescribed boundary function $\vec{u}_0$), then $\vec{u} := \vec{u}_1 + \vec{u}_2$ with the corresponding pressure $p := p_1 + p_2$ solve the problem

$$\begin{aligned} -\Delta\vec{u} + \operatorname{grad} p &= \vec{f}, \text{ and } \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ \vec{u} &= \vec{u}_0, \text{ on } \partial\Omega. \end{aligned}$$

1.1 Aims and scope

I have written this thesis as a part of the requirements of the PhD program in Applied mathematics at the Eötvös Loránd University of Sciences. It is mainly based on my three published papers [53], [54] and [55].

The main part of this work deals with the first kind Stokes problem posed on a planar domain with homogeneous Dirichlet boundary condition along its boundary. Because the problem domain is two-dimensional there is a chance to use the framework of the complex function theory. Hence we consider the problem domain as a proper subdomain of the complex plane $\mathbb{C}$ and we take complex and real valued functions instead of vector and scalar functions, respectively. If particularly the problem domain is simply connected with more than one boundary points, then the use of conformal mapping allows the transformation of the Stokes problem into another problem posed on the unit disc (for it can be chosen as a canonical domain by the well known Riemann mapping theorem). Several properties of the first kind Stokes problem (stable solvability for example) depend strongly on the shape of the problem domain which is, on the other hand, encoded into analytical properties of the univalent function realizing the conformal map between the unit disc and the problem domain. Particulary significant is the possibility to get values or estimates for the value of the so called inf-sup constant of the domain figuring in the condition for stable solvability [25] of the Stokes problem. The aim of the first part of this work was to utilize this possibility for several families of planar domains.

In the second part we deal with various representations of Stokes functions which are the velocity solutions of the homogeneous Stokes equation to prescribed nonzero boundary values. Here we deal also with three-dimensional domains, so we do not use only complex functions.

1.2 Structure of the thesis

In this subsection we give an overview over the structure of this thesis and the new or partially new own results within. Preliminary results are called Proposition throughout this thesis and the origin of these results is given in the references. Own new or partially new results are called lemma, theorem or corollary. If such a result is not new, then it is included with the intetion to demonstrate that our approach gives comparable results to those ones already known. In such a case the reference to the known result is always given in the text before the theorem or in a remark after it. The thesis contains also several remarks and examples, for further specification or utilization of the results, respectively.

The thesis contains two main sections which are splitted into several subsections. Each section begins by clearing the notation used throughout the section and by posing and formulating the specific problem to be dealt with. Definitions and preliminary results are also given. In the succeeding subsections we formulate own or partially own results.

In section 2 we investigate the first kind Stokes problem posed on a plane domain from several aspects with the help of conformal mapping. In the first subsection preliminaries are specified: we define the Schur complement operator of the first kind Stokes problem and the so called Friedrichs operator which is a characteristic operator of the plane domain on which it is supported. We also define the inf-sup constant which is a number depending only on the shape of the problem domain having a decisive role for the stable solvability of the Stokes problem. On the other hand the Friedrichs operator also reflects geometric properties of its supporting domain. There is a simple connecion between these operators which is proved next along a matrix representation of the operators on several domains allowing such a representation. In subsection 2.5 we deal with the question how these operators depend on the domain. Next several spectral properties of the operators are given along with some estimations of important constants (especially the inf-sup constant). In the following three subsections we are concerned with special cases: domains obtainable by polynomial conformal mapping, special families of domains with smooth boundary and domains with corners. Although in section 2 we are mainly dealing with simply connected domains, in subsection 2.11 we touch the problem of multiply connected domains.

In section 3 we omit the complex formalism, because we concern also three-dimensional domains, and give results on the representation of Stokes functions (although also, we point out how these formulae are related to the results of the preceding section in case of planar domains). We also generalize these representation formulae to the case of Naviers equations in linear elasticity.

2 Results via conformal mapping

2.1 Notations, definitions and preliminary results

Let Ω denote throughout this section a domain, i.e. a connected open set, properly contained in $\mathbb{C}$ whose boundary consists of more than one point. If Ω is simply connected (in 2.11 we consider also the multiply connected case), then, by the well known Riemann Mapping Theorem (see for example [37]), it can be mapped conformal (i.e. in such a way that the angle between two differentiable arcs is preserved) onto the open unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ so that an arbitrary point of Ω and a direction through this point correspond, respectively, to the center of D and to the direction of the positiv real axis. (D and ∂D denote in this thesis, respectively, the open unit disc and the unit circle centered at the origin of the complex plane.) We consider only holomorphic functions $g : D \to \Omega$, being the inverse of the Riemann map, which are univalent in D, because then $\Omega = g(D)$ is a schlicht (not self overlapping) domain. Moreover, several geometric properties (boundary regularity, boundedness etc.) of Ω are closely related to analytic properties of its conformal map g (boundary behaviour on ∂D, boundedness etc.). For example Carathéodory's theorem in complex analysis states that if Ω is a simply connected domain of the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ is a Jordan curve (called Jordan domain) then the Riemann map g^{-1} from Ω to D extends continuously to the boundary, moreover, if g maps D conformally onto a Jordan domain Ω, then the boundary $\partial\Omega$ is rectifiable if and only if the derivative g' of the mapping belongs to the Hardy class $H^1(D)$ (defined below in this subsection) , see [19]. If further smoothness conditions are imposed on the boundary of Ω, then we also have more smoothness for the conformal mapping. Therefore we define here some classes of smooth Jordan curves.

Definition 2.1 *A Jordan curve is in the class $C^{n,\alpha}$ for some fixed $n = 1, 2, \ldots$ and $0 < \alpha < 1$, if it has a parametrization $w(t)$, $0 \le t \le 2\pi$ which is n times continuously differentiable and satisfies $w'(t) \neq 0$ and $|w^{(n)}(t_1) - w^{(n)}(t_2)| \le C|t_1 - t_2|^\alpha$.*

The latter condition is actually a Hölder condition with exponent α.

To an appropriate formulation of the considered problems we need – beside the characterization of the problem domain Ω – some function spaces supported on Ω (or D) which we list next.

$L_2(\Omega)$ denotes the space of square integrable functions on Ω with respect to the usual planar Lebesque measure $dA = dxdy$. It is a complete Hilbert space with the inner product of $f, g \in L_2(\Omega)$ defined by $(f, g) := \int_\Omega f\overline{g}dA$. The norm of a function $f \in L_2(\Omega)$ is $\|f\| := (\int_\Omega |f|^2 dA)^{1/2}$.

$L_{2,0}(\Omega) \subseteq L_2(\Omega)$ is the subspace of functions with zero integral over Ω.

$W^{1,2}(\Omega)$ denotes the Sobolev space of functions defined on Ω with generalized derivative in $L_2(\Omega)$ and let $W_0^{1,2}(\Omega)$ be the subspace of $W^{1,2}(\Omega)$ with zero boundary values in the sense of traces on $\partial\Omega$ (see e.g. [1]).

The Hardy spaces of Ω will be denoted by $H^p(\Omega)$ for $p > 0$ (see [19]).

The Bergman space $AL_2(\Omega)$ of complex analytic functions in Ω which belong to $L_2(\Omega)$ is a Hilbert space with a reproducing kernel $K_\Omega(z, \zeta)$ called Bergman kernel of the domain Ω, that is we have for all $f \in AL_2(\Omega)$

$$f(z) = \int_\Omega K_\Omega(z, \zeta) f(\zeta) dA(\zeta). \tag{3}$$

The Bergman kernel is analytic in its first variable, conjugate analytic in the second variable and has the property $K_\Omega(z, \zeta) = \overline{K_\Omega(\zeta, z)}$. The Bergman kernel of the unit disc D is

$$K_D(z, \zeta) = \frac{1}{\pi(1 - z\bar{\zeta})^2},$$

and we have the transformation formula

$$K_D(z, \zeta) = g'(z) K_\Omega(g(z), g(\zeta)) \overline{g'(\zeta)} \tag{4}$$

under the conformal mapping $g : D \to \Omega$ (for further details we refer to [6]).

2.1.1 The Stokes problem and the Schur complement operator

Now we turn to the formulation of the first kind Stokes problem on the plane domain Ω. In classical formulation we look for a velocity function $\vec{u} = (u_1, u_2)^T$ which is twice continuously differentiable in Ω with continuous extension on $\partial\Omega$ and for a corresonding pressure function p which is continuously differentiable in Ω such that for a given $\vec{f}$ which is continuous in Ω we have

$$\begin{aligned} -\Delta\vec{u} + \operatorname{grad} p &= \vec{f}, & (5)\\ \operatorname{div} \vec{u} &= 0 \text{ in } \Omega, & (6)\\ \vec{u} &= 0 \text{ on } \partial\Omega & (7) \end{aligned}$$

Here grad (often denoted also by ∇) and Δ denote the usual gradient and Laplacian of a scalar function and div denotes the usual divergence of a vector function. The Laplacian of a vector function is understood componentwise (if using Cartesian coordinates). Note that, without a normalization, the pressure p is determined only up to an additive constant.

The same problem can be formulated also in variational formulation, i.e. for a given $\vec{f} \in L_2(\Omega)^2$ (which means that each component of $\vec{f}$ is from $L_2(\Omega)$) we look for $\vec{u} = (u_1, u_2)^T \in W_0^{1,2}(\Omega)^2$ (which means that each component of $\vec{u}$ is from $W_0^{1,2}(\Omega)$) and for $p \in L_{2,0}(\Omega)$ such that

$$(\vec{u}, \vec{v})_1 + (-\operatorname{div} \vec{u}, p) = \left(\vec{f}, \vec{v}\right), \text{ for all } \vec{v} \in W_0^{1,2}(\Omega)^2, \tag{8}$$

$$(-\operatorname{div} \vec{u}, q) = 0, \text{ for all } q \in L_{2,0}(\Omega). \tag{9}$$

Here – based on the scalar product $(\cdot, \cdot)$ of $L_2(\Omega)$ – we use the scalar products $(\vec{u}, \vec{v}) := (u_1, v_1) + (u_2, v_2)$ and

$$(\vec{u}, \vec{v})_1 := \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dA$$

for vector functions from the spaces $L_2(\Omega)^2$ and $W_0^{1,2}(\Omega)^2$, respectively. Introducing also the functional

$$\beta^2(\vec{u}, p) := \frac{(\operatorname{div} \vec{u}, p)^2}{(\vec{u}, \vec{u})_1 (p, p)}$$

we can write the inf-sup condition (or LBB-condition after Brezzi [8], Babuška [5] and Ladyzhenskaya [32]) assuring the stable solvability of (8) as

$$\inf_{0 \neq p \in L_{2,0}} \sup_{0 \neq u \in W_0^{1,2}(\Omega)^2} \frac{(\operatorname{div} u, p)^2}{(u, u)_1 (p, p)} = \beta_0^2(\Omega) > 0, \tag{10}$$

where $\beta_0(\Omega) > 0$ is the so-called inf-sup constant depending only on the shape of Ω. Actually we have further $0 < \beta_0(\Omega) \leq 1$, see [45]. For the fact that (10) holds under appropriate conditions imposed on Ω, one usually refers to [36]. A detailed investigation of this condition is contained in [9] and [25].

Decisive for the solution of the Stokes problem is the Schur complement operator which is defined by

$$\mathcal{S} = \operatorname{div} \Delta_0^{-1} \operatorname{grad}, \tag{11}$$

where Δ_0 denotes the vector Laplace operator corresponding to homogeneous Dirichlet boundary values. The operator (11) has been examined in [15], [24] and [45]. The eigenvalues of the Schur complement operator, i.e.

$$\Delta u = \operatorname{grad} p; \operatorname{div} u = \lambda p \text{ in } \Omega \text{ ; and } u = 0 \text{ on } \partial\Omega,$$

are known to lie in $[0, 1]$ and are closely related to a decomposition of the space $L_2(\Omega)$ into three orthogonal subspaces

$$L_2(\Omega) = P_0 \oplus P_1 \oplus P_\beta \tag{12}$$

with orthogonality in the sense of its scalar product, see [45]. These subspaces

$$P_0 := \ker \operatorname{grad}, P_1 := \operatorname{div} \ker \operatorname{rot} = \operatorname{div} V_1 \text{ and } P_\beta = \operatorname{div} V_\beta,$$

where P_0 is the one dimensional space of constant functions and P_β consists of harmonic functions, are related to a decomposition

$$W_0^{1,2}(\Omega)^2 = V_0 \oplus V_1 \oplus V_\beta$$

orthogonal with respect to the scalar product $(\cdot, \cdot)_1$, where

$$\begin{aligned} V_0 &:= \ker \operatorname{div} = \left\{ \vec{v} \in W_0^{1,2}(\Omega)^2 : \operatorname{div} \vec{v} = 0 \right\}, \\ V_1 &:= \ker \operatorname{rot} = \left\{ \vec{v} \in W_0^{1,2}(\Omega)^2 : \operatorname{rot} \vec{v} = 0 \right\}, \end{aligned}$$

and $\operatorname{rot} \vec{v}$ denotes the scalar $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$. The latter decomposition has been derived in [14] and [49], where the third orthogonal subspace V_β has been also characterized: it consists of biharmonic vector functions (see also Lemma 1 in [45]). Both are called Crouzeix-Velte decompositions. We further have, according to [25] and [45], that the operator $\Delta_0^{-1} \operatorname{grad}$ is an isomorphism from P_1 and P_β onto V_1 and V_β, respectively, whereas div is an isomorphism from V_1 and V_β onto P_1 and P_β, both conserving orthogonalities. [45] contains also

$$P_1 = \operatorname{div} V_1 = \operatorname{rot} V_0 \perp \operatorname{rot} V_\beta = \operatorname{div} V_\beta = P_\beta.$$

The eigenspaces to the eigenvalues of the Schur complement operator $\mathcal{S}$ are connected to the subspaces of the Crouzeix-Velte decomposition of $L_2(\Omega)$ and also to the inf-sup constant. The zero and unit eigenvalue correspond, respectively, to constant pressures (P_0) and pressures from P_1 which are the divergence of rotation-free velocities, so the restriction $\mathcal{S}_{|P_1}$ is the identity operator. The third subspace P_β is spanned by eigenfunctions of $\mathcal{S}$ belonging to eigenvalues which lie in $(0, 1)$, see [45]. The square of the inf-sup constant $\beta_0(\Omega)$ is the smallest among the latter eigenvalues; together with λ also $1-\lambda$ is an eigenvalue of the restriction $\mathcal{S}_{|P_\beta}$ (see [50]) and we have for the eigenvalues of this restricted operator (see [15] and [45])

$$\beta_0^2(\Omega) \leq \lambda(\mathcal{S}_{|P_\beta}) \leq 1 - \beta_0^2(\Omega). \tag{13}$$

The eigenvalue and eigenspace structure of the Schur complement operator (especially the inf-sup constant) are very important also for the numerical solution of the Stokes problem: for stability and error estimates connected to the Stokes problem and also for the iterative solution of discretized Stokes and Navier-Stokes problems. The utilization of this knowledge for the acceleration of iterative methods has been investigated in [46]. See also [18] and [48].

Regardless of its importance, explicit values of the inf-sup constant for specific domains are known only in a few cases: for the circle, the annulus [10], and the ellipse, see [26], and for an infinite strip - assuming periodicity along the strip [34], and, in the three-dimensional case, for the sphere [49]. Some lower and upper bounds for inf-sup constants of several domains are derived in [45]. Results for channel and plate domains are derived in [16] and [17]. [13] and [52] contain related work.

2.1.2 The Friedrichs operator

Another operator which plays a role in this thesis is the Friedrichs operator which was implicitly introduced by K. Friedrichs [21] intended for applications to planar elasticity.

The Friedrichs operator of the domain Ω is defined by

$$\mathcal{F} = \mathcal{P} \circ \mathcal{C} : AL_2(\Omega) \to AL_2(\Omega), \tag{14}$$

where $\mathcal{P}$ is the orthogonal projection – called Bergman projection – of $L_2(\Omega)$ onto $AL_2(\Omega)$ and $\mathcal{C}$ denotes the conjugacy operator, i.e. $\mathcal{C}f := \bar{f}$. It can be expressed also as an integral transformation using the Bergman kernel

$$\mathcal{F} : AL_2(\Omega) \to AL_2(\Omega), \quad \mathcal{F}(f)(z) = \int_\Omega K_\Omega(z,\zeta)\overline{f(\zeta)}dA(\zeta). \tag{15}$$

The Friedrichs operator is conjugate-linear, i.e.

$$\mathcal{F}(\lambda f + \mu g) = \bar{\lambda}\mathcal{F}(f) + \bar{\mu}\mathcal{F}(g) \text{ for } f, g \in AL_2(\Omega) \text{ and } \lambda, \mu \in \mathbb{C},$$

while the operator

$$\mathcal{T} := \mathcal{C} \circ \mathcal{F} \tag{16}$$

is C-symmetric in the sense of [23], i.e.

$$\mathcal{T} = \mathcal{C}\mathcal{T}^*\mathcal{C},$$

where $\mathcal{T}^*$ denotes the adjoint of the operator $\mathcal{T}$ on $AL_2(\Omega)$ with respect to its scalar product $(\cdot,\cdot)$.

It can be easily computed that the Friedrichs operator of the unit disc has rank one [21], but for more general domains it may not even be compact. The compactness of $\mathcal{F}$ in terms of the conformal map $g : D \to \Omega$ has been characterized as follows

Proposition 2.2 ***(Lin-Rochberg, [33])*** *Let Ω be a proper, simply connected domain in $\mathbb{C}$ and let g be the conformal mapping of the unit disc D onto Ω. Then $\mathcal{F}$ is compact if and only if $\mathcal{P}_D(g'/\overline{g'}) \in B_0(D)$, where $B_0(D)$ is the little Bloch space which consists of holomorphic functions f on D such that $(1-|z|^2)f'(z) \to 0$ as $|z| \to 1$ and where $\mathcal{P}_D$ denotes the Bergman projection connected with D.* □

$\mathcal{F}$ is the underlying operator of the following eigenvalue problem: find $f \in AL_2(\Omega)$ and the corresponding $\mu \in \mathbb{C}$ such that

$$\int_\Omega fg dA = \mu \int_\Omega \bar{f} g dA, \text{ for all } g \in AL_2(\Omega), \tag{17}$$

which was studied in [21], because

$$(f, \mathcal{F}g) = (g, \mathcal{F}f) = \int_\Omega fg dA \text{ for } f, g \in AL_2(\Omega),$$

see Theorem 4 in [21]. This eigenvalue problem is on the other hand connected to the so-called Friedrichs inequality which reads as follows.

Proposition 2.3 ***(Friedrichs, [21])*** *Let the boundary of the bounded domain Ω have piecewise smooth boundary, with finitely many (n) corners with interior angles α_k, $0 < \alpha_k \leq 2\pi$, $1 \leq k \leq n$. Then there exists a positive constant $\gamma_\Omega < 1$ such that*

$$\left| \int_\Omega f^2 dA \right| \leq \gamma_\Omega \int_\Omega |f|^2 dA \text{ for all } f \in AL_{2,0}(\Omega). \tag{18}$$

□

So we have $\|\mathcal{F}(f)\| \leq \|f\|$ for $f \in AL_2(\Omega)$ and in case of a bounded domain $\mathcal{F}(1) = 1$, where 1 also denotes the function identically 1. For the square of the Friedrichs operator, which is $\mathbb{C}$-linear, we obtain

$$(f, \mathcal{F}^2 g) = (\mathcal{F}g, \mathcal{F}f) = \overline{(\mathcal{F}f, \mathcal{F}g)} = \overline{(g, \mathcal{F}^2 f)} = (\mathcal{F}^2 f, g)$$

and especially

$$(f, \mathcal{F}^2 f) = (\mathcal{F}f, \mathcal{F}f),$$

whence $\mathcal{F}^2$ is self-adjoint on $AL_2(\Omega)$ satisfying $0 \leq \mathcal{F}^2 \leq \mathcal{I}$, where $\mathcal{I}$ denotes the identity operator. Therefore the square root of $\mathcal{F}^2$ can also be defined, which is called the modulus of $\mathcal{F}$ and is denoted by $|\mathcal{F}|$.

According to [21], if the operator $\mathcal{F}^2$ of a bounded domain Ω is compact, then it has countable many eigenvalues, its first eigenvalue equals 1 (the eigenspace spanned by the constant 1 function) while all the other eigenvalues are less than 1 so that the biggest among them is the square of the constant γ from Proposition 2.3.

In case $\mathcal{F}^2$ is not compact there also appears a continuous spectrum containing the values $\left|\frac{\sin\alpha_k}{\alpha_k}\right|$ for $0 \le k \le n$ using the notations of Proposition 2.3 along with $\alpha_0 = \pi$.

For an extensive study of further properties of the Friedrichs operator we refer to [23], [33], [39], [40] and the references given therein.

2.2 Connection between the problems

In this subsection the connection between the Friedrichs and the Schur complement operators is investigated. To do this we need to unify the notations used in the preceding subsections. Instead of the real vector variable $\vec{x} = (x_1, x_2)^T$ we use the complex variable $z = x_1 + x_2 i$. Vector valued functions $\vec{u}(\vec{x}) = (u_1(\vec{x}), u_2(\vec{x}))^T$ for the velocity are replaced by complex valued functions $u(z) = u_1(z) + u_2(z)i$. Scalar valued functions (i.e. for the pressure) are replaced by real valued ones. Further we use (see [37])

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \text{ and } \partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Hence for a complex valued function $u = u_1 + iu_2$ the divergence and rotation are given by

$$\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 2\operatorname{Re}\partial_z u, \text{ and } \operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 2\operatorname{Im}\partial_z u.$$

For a real valued function p the (complex valued) gradient is

$$\operatorname{grad} p = \frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y} = 2\partial_{\bar{z}} p.$$

The Laplacian is expressed as $\Delta = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$.

Next, the boundary of Ω is supposed to be sufficiently regular in the sense that the space $W^{1,2}(\Omega)$ has traces in $L_2(\partial\Omega)$. This is certainly fulfilled for domains with piecewise C^2 regular boundary [1]. To establish a connection between $\mathcal{S}$ and $\mathcal{F}$ we need

Proposition 2.4 ***(Corollary 2.5 in [40])*** *Let the boundary of the plane domain Ω be sufficiently regular in the sense that the space $W^{1,2}(\Omega)$ has traces in $L_2(\partial\Omega)$. Set $f \in H^2(\Omega)$. The solution to the Dirichlet problem:*

$$\Delta u = 0 \text{ in } \Omega; \quad u(\zeta) = \overline{\zeta} f(\zeta), \text{ for } \zeta \in \partial\Omega,$$

is $u(z) = \overline{G(z)} + h(z)$*, where* $G \in H^2(\Omega)$ *is the primitive function of* $\mathcal{F}(f)$ *and the function* $h \in H^2(\Omega)$ *is defined by*

$$h(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{\zeta} f(\zeta) - \overline{G(\zeta)}}{\zeta - z} d\zeta. \quad \square$$

□

Using this representation we reestablish the following theorem already announced in [13] where, however, the smoothness of the boundary has not been specified.

Theorem 2.5 *Let* Ω *be a domain with a boundary as in Proposition 2.4. Then we have*

$$2\mathcal{S} = \mathcal{I} - \mathcal{C} \circ \mathcal{F} \tag{19}$$

for the Friedrichs operator $\mathcal{F}$ *and the Schur complement operator* $\mathcal{S}$ *of the domain.*

Proof. Assume $f \in H^2(\Omega)$ and set $u_0(z) = \frac{1}{2} z \overline{f(z)}$, $p_R(z) = 2 \operatorname{Re} f(z)$. There follows $\Delta u_0 = \nabla p_R$ in Ω (see [52]) and

$$\mathcal{S} p_R(z) = \operatorname{div} u_0(z) = 2 \operatorname{Re} \partial_z u_0(z) = \operatorname{Re} \overline{f(z)} = \frac{1}{2} p_R,$$

but, in general, u_0 does not fulfil the homogeneous boundary condition. Let us solve the Dirichlet problem

$$\Delta H = 0 \text{ in } \Omega, H(z) = \overline{z} f(z) \text{ for } z \in \partial\Omega.$$

By Proposition 2.4 we have $H = \overline{G} + h$, where

$$G'(z) = \mathcal{F}(f)(z) = \int_\Omega K_\Omega(z, \zeta) \overline{f(\zeta)} dA(\zeta).$$

Now $u = u_0 - \frac{1}{2}\overline{H}$ satisfies $\Delta u = \nabla p_R$ in Ω with homogeneous boundary values. Further we have for the divergence

$$\operatorname{div} u(z) = \operatorname{div} u_0(z) - \operatorname{Re} \partial_z \overline{H(z)} = \frac{1}{2} p_R(z) - \operatorname{Re} G'(z),$$

which gives

$$\mathcal{S} p_R(z) = \frac{1}{2} p_R(z) - \operatorname{Re} \mathcal{F}(f)(z). \tag{20}$$

Analogously, with $-if$ instead of f, we obtain for $p_I = 2 \operatorname{Im} f$

$$\mathcal{S} p_I(z) = \frac{1}{2} p_I(z) + \operatorname{Im} \mathcal{F}(f)(z). \tag{21}$$

Combining the equalities (20) and (21) gives for $2f = p_R + ip_I$

$$2\mathcal{S}(f)(z) = f(z) - \overline{\mathcal{F}(f)(z)}, \tag{22}$$

for $f \in H^2(\Omega)$. By [7] the space $H^2(\Omega)$ is a dense subspace of $AL_2(\Omega)$ for domains involved in the theorem, hence (22) remais valid for $f \in AL_2(\Omega)$, which proves the theorem. □

Remark 2.6 Equations (20) and (21) show that the deviation of the operator $\mathcal{S}$ from $\frac{1}{2}\mathcal{I}$ is due to the boundary of the domain. □

Remark 2.7 Equation (19) holds only if we apply it on a function $f \in AL_2(\Omega)$. Hence equations (20) and (21) are valid in case p_R and p_I are harmonic (as being the real and imaginary parts of an analytic function). So (19) is valid in the following sense: we take the restriction $\mathcal{S}_{|P_0 \oplus P_\beta}$ (see (12)) of the Schur complement operator $\mathcal{S}$ defined by (11), we extend its domain from real valued (pressure)functions to complex valued functions from $AL_2(\Omega)$ by equations (20) and (21) and the so obtained operator we denote again by $\mathcal{S}$. As a matter of fact, we can so examine the Schur complement operator only on $P_0 \oplus P_\beta$ via the Friedrichs operator (and hence via conformal mapping). □

Remark 2.8 This preceding proof of Theorem 2.5 can be found in [54]. Another proof involving the conformal map g of the unit disc D onto Ω is essentially contained in [53]. This proof contains also matrix representations of the operators which we show in subsection 2.4. □

Remark 2.9 Observe that by definition (16) of $\mathcal{T}$ and Theorem 2.5 there also follows

$$2\mathcal{S} = \mathcal{I} - \mathcal{T}.$$

□

With the help of Theorem 2.5 several known properties of the Friedrichs operator of a simply connected planar domain with sufficiently smooth boundary carry over to the Schur complement operator of the same domain and vice versa. For example compactness and spectral properties depending on the shape of the domain. Theorem 2.5 opens the possibility to determine or estimate the inf-sup constant (10) with the help of the conformal mapping of the domain, because it is connected to the Bergman kernel which is on the other hand connected to the Bergman projection involved in the definition (14) of the Friedrichs operator. We calculate several examples in the subsections 2.7, 2.8 and 2.9.

2.3 Usage of conformal mapping

The potential utility of the usage of conformal mapping in the study of the operators figuring in Theorem 2.5 comes with the transformation property (4) of the Bergman kernel which is involved in the definition (15) of the Friedrichs operator.

Let g denote the conformal mapping of the unit disc (of the z-plane) onto the domain Ω (in the w-plane), i.e. $w = g(z)$. From (15) described with the variable w, i.e.

$$\mathcal{F}(f)(w) = \int_\Omega K_\Omega(w, \omega)\overline{f(\omega)}dA(\omega),$$

there follows by change of the variable (i.e. $w = g(z)$, $\omega = g(\zeta)$ and $dA(\omega) = |g'(\zeta)|^2 dA(\zeta)$)

$$\begin{aligned}\mathcal{F}(f)(g(z)) &= \int_D \frac{1}{g'(z)} K_D(z,\zeta) \frac{1}{\overline{g'(\zeta)}} \overline{f(g(\zeta))} |g'(\zeta)|^2 dA(\zeta) \\ &= \frac{1}{g'(z)} \int_D K_D(z,\zeta) \frac{g'(\zeta)}{\overline{g'(\zeta)}} \overline{p(\zeta)} dA(\zeta),\end{aligned}$$

where we have also set $p(z) := f(g(z))g'(z)$. This imples also $\int_\Omega |f|^2 dA = \int_D |p|^2 dA$, thus the L_2 norms of f and p on Ω and D, respectively, are equal. Now we can define

$$\mathcal{F}_D(p)(z) := g'(z)\mathcal{F}(f)(g(z)) = \int_D K_D(z,\zeta) \frac{g'(\zeta)}{\overline{g'(\zeta)}} \overline{p(\zeta)} dA(\zeta). \qquad (23)$$

So the operator $\mathcal{F}$ defined on $AL_2(\Omega)$ is unitarily equivalent to $\mathcal{F}_D$ defined on $AL_2(D)$. In fact, this unitary equivalence can also be found in the proof of Proposition 2.2, see [33].

The solutions of the homogeneous momentum equation (5) (with $\vec{f} = 0$) of the Stokes problem can also be studied with help of conformal mapping. The main tool is the following proposition due to Kratz and Peyerimhoff (see [31] and also [52]) which reads – in our notation – essentially as follows.

Proposition 2.10 *Let Ω be a simply connected plane domain and f holomorphic in Ω. Then*

$$U(w) = \frac{1}{2} w\overline{f(w)} + V_1(w) + \overline{V_2(w)} \text{ and } P(w) = 2\,\mathrm{Re}\, f(w)$$

fulfil the homogeneous momentum equation $\Delta U(w) = \mathrm{grad}\, P(w)$ on Ω with holomorphic functions V_1 and V_2. Moreover, if we have $f(w) = -2V_1'(w)$, then $\mathrm{div}\, U(w) = 0$. □

Remark 2.11 Observe that choosing V_1 and V_2 such that U fulfils the homogeneuos boundary condition on $\partial\Omega$ (i.e $U(w) = 0$ for $w \in \partial\Omega$) implies that

$$\begin{aligned} \mathcal{S}P(w) &= \operatorname{div} U(w) = 2\operatorname{Re}\partial_w U(w) \\ &= \operatorname{Re} f(w) + 2\operatorname{Re} V_1'(w) = \frac{1}{2}P(w) + 2\operatorname{Re} V_1'(w). \end{aligned}$$

□

Now, if we set $u = U \circ g$, i.e.

$$\begin{aligned} u(z) &:= U(g(z)) = \frac{1}{2}g(z)\overline{f(g(z))} + V_1(g(z)) + \overline{V_2(g(z))} \\ &= \frac{1}{2}g(z)\frac{\overline{p(z)}}{\overline{g'(z)}} + v_1(z) + \overline{v_2(z)} \end{aligned}$$

using again the conformal mapping $w = g(z)$, then we can transform the statement of Proposition 2.10 into another statement formulated with holomorphic functions on the unit disc D. This is Lemma 3.7 in [52] which we reformulate with the notations of this section:

Proposition 2.12 *Let g be the conformal map of the unit disc D onto Ω, p holomorphic on D. The function $u : D \to \mathbb{C}$, for which the transformed function $U = u \circ g^{-1}$ fulfils the homogeneous momentum equation*

$$\Delta U = \mathit{grad}\, P$$

on Ω with the corresponding pressure

$$P = 2\,\mathit{Re}\, f, \ \ \mathit{where}\ f = \frac{p \circ g^{-1}}{g' \circ g^{-1}},$$

can be represented by the formula

$$u(z) = \frac{1}{2}g(z)\overline{\left(\frac{p(z)}{g'(z)}\right)} + v_1(z) + \overline{v_2(z)} \tag{24}$$

with holomorphic functions v_1 and v_2 on D. Further u has the divergence

$$\mathit{div}(u \circ g^{-1}) \circ g = \mathit{Re}\,\frac{p}{g'} + 2\,\mathit{Re}\,\frac{v_1'}{g'}. \tag{25}$$

□

Remark 2.13 The equation (25) shows also that

$$\mathcal{S}P(g(z)) = \operatorname{Re} \frac{p(z)}{g'(z)} + 2 \operatorname{Re} \frac{v_1'(z)}{g'(z)} \text{ for } z \in D.$$

□

The correspondence of the operators in Theorem 2.5 can be established also on the basis of these propositions in which the conformal mapping is explicitly involved. As we see, the crucial point is to find appropriate holomorphic functions v_1 and v_2 such that the transformed velocity function $U = u \circ g^{-1}$ fulfils the homogeneuos boundary condition on $\partial\Omega$ (i.e $U(w) = 0$ for $w \in \partial\Omega$). But this is equivalent to $u(z) = 0$ for $z \in \partial D$ as long as the conformal mapping is continuously extendable to the boundary constituting a bijective mapping between $\partial\Omega$ and ∂D. This is however possible for rather general domains Ω (for example if $\partial\Omega$ is a closed Jordan curve), see e.g. [19] for more details.

Lemma 2.14 *Let g be the conformal mapping of D onto Ω and let $p(z)$ be holomorphic on D such that the function*

$$u_0(z) := \frac{1}{2} g(z) \overline{\left(\frac{p(z)}{g'(z)} \right)}$$

has boundary values in $L_2(\partial D)$. Then there are functions $v_1, v_2 \in H^2(D)$ such that (24) *has zero boundary values on ∂D.*

Proof. The needed holomorphic function v_1 can be found by recovering it from the projection of the boundary values of u_0 onto the $H^2(\partial D)$ (as a subspace of $L_2(\partial D)$), see [19]. v_2 is recovered similarly. We have for $z \in D$

$$v_1(z) := \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1}{1 - z\bar{\zeta}} \left(-\frac{1}{2} g(\zeta) \overline{\frac{p(\zeta)}{g'(\zeta)}} \right) |d\zeta|, \tag{26}$$

$$v_2(z) := \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1}{1 - z\bar{\zeta}} \left(-\frac{1}{2} \overline{g(\zeta)} \frac{p(\zeta)}{g'(\zeta)} - \overline{v_1(\zeta)} \right) |d\zeta|. \quad \square \tag{27}$$

Now, the derivative of v_1 figuring in (25) is expressed as

$$\begin{aligned} v_1'(z) &= -\frac{1}{4\pi} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(1 - z\bar{\zeta})^2} g(\zeta) \overline{\frac{p(\zeta)}{g'(\zeta)}} |d\zeta| \\ &= \frac{1}{4\pi i} \int_{|\zeta|=1} \frac{1}{(1 - z\bar{\zeta})^2} g(\zeta) \overline{\frac{p(\zeta)}{g'(\zeta)}} d\bar{\zeta} \\ &= -\frac{1}{2\pi} \int_D \frac{1}{(1 - z\bar{\zeta})^2} \frac{g'(\zeta)}{\overline{g'(\zeta)}} \overline{p(\zeta)} dA(\zeta). \end{aligned}$$

Therefore the equation

$$v_1'(z) = -\frac{1}{2}\mathcal{F}_D(p)(z) \tag{28}$$

follows with the operator $\mathcal{F}_D$ defined in (15). If we divide by g' and we transform the latter result from the unit disc D onto Ω, then we obtain $V_1 = -\frac{1}{2}\mathcal{F}(f)$, from which we easily have (20). The same calculation with $-if$ instead of f gives (21) and hence we reproved Theorem 2.5 without explicitely using Proposition 2.4.

Remark 2.15 The mapping function g must be certainly smooth enough such that we can use Lemma 2.14 to obtain the function v_1 so that its derivative lies in $AL_2(D)$ (see (28)) for $p \in AL_2(D)$. By [28] it is certainly sufficient that g' is continuous up to the boundary ∂D of the unit disc, and its boundary function is Hölder continuous with an exponent $0 < \alpha < 1$, which holds if $\partial\Omega$ is a $C^{1,\alpha}$ curve. Hence we have substituted the smoothness condition imposed on Ω in Proposition 2.4 with another smoothness condition imposed on the corresponding conformal mapping. □

2.4 Matrix representation

The main idea of the above alternate proof of Theorem 2.5 originates in [52], where it is made only for polynomial mapping. For the case of non polynomial mapping it was generalized in [53] using however not the same formulae (26) and (27) but their equivalents in a certain matrix formulation which we show in this subsection. The investigation of the structure of the (finite or infinite) matrix representant of the operators reveals several properties of the operators in dependence on the problem domain (value and multiplicity of eigenvalues for example).

For the sake of correctness we must define an appropriate space of infinite sequences which an infinite matrix $\mathcal{M}$ acts on. Let us consider the spaces of sequences

$$\ell_{(2,-\alpha)} := \left\{ p \mid p = (p_0, p_1, p_2, \dots)^T \text{ fulfilling } \sum_{n=0}^{\infty} \frac{n!\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)} |p_n|^2 < \infty \right\}$$

for some $\alpha > 0$, where Γ denotes the standard Γ-function from complex analysis. $\ell_{(2,-\alpha)}$ is a Hilbert space with the usual scalar product

$$(p,q)_{-\alpha} := \sum_{n=0}^{\infty} \frac{n!\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)} p_n \bar{q}_n,$$

for $p, q \in \ell_{(2,-\alpha)}$, the norm of p being defined by $\|p\|_{-\alpha} := \sqrt{(p,p)_{-\alpha}}$. If we set for $\alpha > 0$ and $p, q \in \ell_{(2,-\alpha)}$

$$p(z) := \sum_{n=0}^{\infty} p_n z^n \text{ and } q(z) := \sum_{n=0}^{\infty} q_n z^n (z \in D),$$

then the scalar product of $\ell_{(2,-\alpha)}$ can be expressed by the integral

$$(p,q)_{-\alpha} = \alpha \int_D p(z)\overline{q(z)}(1-|z|^2)^{\alpha-1} dA(z).$$

Hence $\ell_{(2,-\alpha)}$ is naturally isometric with the Hilbert space of analytic functions on the unit disc D for which this integral is finite (the so-called weighted Bergman spaces). Now by setting especially $\alpha := 1$ we obtain

$$(p_0, p_1, p_2, \dots)^T \in \ell_{(2,-1)} \Longleftrightarrow \sum_{n=0}^{\infty} p_n z^n \in AL_2(D).$$

On this basis we can introduce a matrix formalism which comes into scope if we use Taylor series of v_1 and v_2 instead of the integrals (26) and (27). We assume that the conformal mapping of D onto Ω is given by the series

$$g(z) = \sum_{m=0}^{\infty} a_m z^m, \tag{29}$$

where $g' \neq 0$ in D, so that we can set

$$\frac{1}{g'(z)} = \sum_{\ell=0}^{\infty} b_\ell z^\ell. \tag{30}$$

Lemma 2.16 *Let Ω be a simply connected plane domain. Let the bijective conformal map $D \to \Omega$ be of the form* (29) *which converges on an open neighbourhood of $\bar{D}$. Suppose further that the reciprocal of its derivative is given by* (30). *Let be $p(z) = \sum_{n=0}^{\infty} p_n z^n \in H^2(D)$. The function u given by the representation* (24) *has zero boundary values on ∂D if*

$$v_1(z) = -\frac{1}{2} \sum_{k=1}^{\infty} \left\{ \sum_{m=k}^{\infty} a_m \left(\sum_{\ell=0}^{m+k} \bar{b}_{m+k-\ell} \bar{p}_\ell \right) \right\} z^k, \tag{31}$$

$$\overline{v_2(z)} = -\frac{1}{2} \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{\infty} a_m \left(\sum_{\ell=0}^{m+k} \bar{b}_{m+k-\ell} \bar{p}_\ell \right) \right\} \bar{z}^k. \tag{32}$$

Proof. Let us substitute the series expansions (29), (30) and of the function p into (24). On rearrangement of the series we obtain

$$\begin{aligned} u(z) &= \frac{1}{2} g(z) \frac{\overline{p(z)}}{\overline{g'(z)}} + v_1(z) + \overline{v_2(z)} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{\infty} a_m \left(\sum_{\ell=0}^{m+k} \bar{b}_{m+k-\ell} \bar{p}_\ell \right) z^m \bar{z}^{m+k} \right\} + v_1(z) + \overline{v_2(z)} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left\{ \sum_{m=k}^{\infty} a_m \left(\sum_{\ell=0}^{m-k} \bar{b}_{m-k-\ell} \bar{p}_\ell \right) z^m \bar{z}^{m-k} \right\} + v_1(z) + \overline{v_2(z)}. \end{aligned}$$

On ∂D we have $z\bar{z} = 1$, whence u has zero boundary values if we define v_1 and v_2 by (31) and (32). The convergence of the power series in this proof is a consequence of a result in [37]: if the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ both converge for $|z| < R$, then the series $C(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{\ell=0}^{n} a_\ell b_{n-\ell}$, also converges for $|z| < R$ and $C(z) = A(z)B(z)$. □

Remark 2.17 The assumption on the mapping g in Lemma 2.16 implies that Ω has an analytic boundary. In this case we can rearrange the series for u in the proof. This strong assumption could be however avoided so that we have the result of Lemma 2.16 also for other more general domains if we can prove that the series in that case are also convergent and define functions which have appropriate boundary values. □

For the derivative of the holomorphic function v_1 from Lemma 2.16 we obtain the series expansion

$$v_1'(z) = -\frac{1}{2} \sum_{k=0}^{\infty} (k+1) \left(\sum_{\ell=0}^{\infty} \left(\sum_{m=\ell+k+1}^{\infty} a_m \bar{b}_{m-(\ell+k+1)} \right) \bar{p}_\ell \right) z^k, \tag{33}$$

from which – comparing with (28) – there follows

$$\mathcal{F}_D(p)(z) = \sum_{k=0}^{\infty} (k+1) \left(\sum_{\ell=0}^{\infty} \left(\sum_{m=\ell+k+1}^{\infty} a_m \bar{b}_{m-(\ell+k+1)} \right) \bar{p}_\ell \right) z^k.$$

This constitutes a correspondence between the coefficients of the series expansions for p and $\mathcal{F}_D(p)$. The coefficients of the expansion for $\mathcal{F}_D(p)(z)$ can be obtained from the vector

$$p := (p_0, p_1, p_2, \dots)^T$$

of the coefficients from the series for $p(z)$ by multiplication with the infinite matrix

$$\mathcal{M} = (s_{k,\ell})_{k,\ell=0}^{\infty}. \tag{34}$$

The entries of $\mathcal{M}$ are $s_{k,\ell} := (k+1)s_{k+\ell}$, where the quantities

$$s_n := \sum_{k=0}^{\infty} a_{n+k+1}\bar{b}_n \tag{35}$$

are taken from the series expansions (29) and (30). The same quantity formulated as an integral is

$$s_n = \frac{1}{\pi}\int_D \overline{z}^n \frac{g'(z)}{\overline{g'(z)}} dA(z). \tag{36}$$

Hence the conjugate linear mapping

$$\mathcal{F}_D : AL_2(D) \to AL_2(D), \quad p \mapsto \mathcal{F}_D(p)$$

has the (infinite) matrix representation

$$\mathcal{MC} : \ell_{(2,-1)} \to \ell_{(2,-1)}, \quad p \mapsto \mathcal{MC}p.$$

Remark 2.18 Suppose that the conformal mapping g is such that the quantities (35) are finite. In that case (36) also applies and we have the estimate

$$|s_n| \leq \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \rho^n \rho d\rho d\theta = \frac{2}{n+2},$$

and therefore $|s_{k,\ell}| \leq \frac{2(k+1)}{k+\ell+2} \leq 2$ for $k, \ell \geq 0$. Hence the matrix representant $\mathcal{M}$ of $\mathcal{F}_D$ has well defined entries which are, by their absolute value, at most 1. □

Remark 2.19 There are domains given by the corresponding function g for which the series defining the quantities (35) are divergent. Such an example is the Koebe function

$$g(z) = \frac{z}{(1-z)^2},$$

which maps D onto the slit plane. □

Remark 2.20 Finally, we formulate a condition for the conformal mapping g so that the matrix representation in this section is possible. A sufficient condition is certainly that the derivative of the conformal mapping g is continuous on an open neighbourhood of the closed unit disc (thus the boundary of the domain is analytic). But this condition is surely not necessary. To achive other conditions let us introduce additional spaces of infinite sequences (see [27]). Set $\alpha \geq 0$ and

$$\ell_{(2,\alpha)} := \left\{ p \mid p = (p_0, p_1, p_2, \dots)^T \text{ with } \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} |p_n|^2 < \infty \right\}.$$

The scalar product of $\ell_{(2,\alpha)}$ (similar to that of $\ell_{(2,-\alpha)}$) is defined by

$$(p,q)_\alpha := \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} p_n \bar{q}_n .$$

As the spaces $\ell_{(2,-\alpha)}$ correspond to the weighted Bergman spaces, so correspond $\ell_{(2,\alpha)}$ to other spaces of holomorphic functions on D. In case $\alpha = 0$ the space $\ell_{(2,0)}$ corresponds to the Hardy space $H^2(D)$. For $\alpha = 1$ the space $\ell_{(2,1)}$ corresponds to the Dirichlet space. The spaces $\ell_{(2,\alpha)}$ are dual to $\ell_{(2,-\alpha)}$ for $\alpha > 0$ in the sense

$$|(p,q)_0|^2 \leq (p,p)_{-\alpha}(q,q)_\alpha ,$$

for $p \in \ell_{(2,-\alpha)}$ and $q \in \ell_{(2,\alpha)}$. The quantities (35) are

$$s_n := \left(S^{n+1}a, b\right)_0 ,$$

where the vector $a := (a_0, a_1, \dots)^T$ is built from the coefficients of g (see (29)), b from (30) and S denotes the left-shift

$$S(a_0, a_1, \dots)^T = (a_1, a_2, \dots)^T .$$

Therefore, we have either

$$|s_n|^2 = \left|(S^{n+1}a, b)_0\right|^2 \leq (a,a)_\alpha (b,b)_{-\alpha} \text{ or } |s_n|^2 = \left|(S^{n+1}a, b)_0\right|^2 \leq (a,a)_{-\alpha}(b,b)_\alpha .$$

Hence, if the coefficients of the mapping function g and that of the reciprocal of its derivative satisfy either

$$a \in \ell_{(2,\alpha)} \text{ and } b \in \ell_{(2,-\alpha)} \quad \text{or} \quad a \in \ell_{(2,-\alpha)} \text{ and } b \in \ell_{(2,\alpha)} ,$$

then the quantites (35) are finite. □

Along with Theorem 2.5 we obtained in this section the following

Corollary 2.21 *The Friedrichs operator $\mathcal{F}$ and Schur complement operator $\mathcal{S}$ of the simply connected domain $\Omega = g(D)$ with appropriately smooth conformal mapping g are, respectively, unitary equivalent to the conjugate linear and complex linear operators*

$$\mathcal{MC} \text{ and } \frac{1}{2}(\mathcal{I} - \mathcal{CMC}) ,$$

which are defined by the infinite matrix $\mathcal{M}$.

2.5 Domain dependence

In this section the dependence of the studied operators on the problem domain is investigated. We make use of the unitary equivalence between $\mathcal{F}$ and $\mathcal{F}_D$.

We are given two simply connected domains Ω and $\tilde{\Omega}$ with the corresponding conformal mappings g and $\tilde{g}$, respectively. Let the function η be such that $\tilde{g} = \eta \circ g$. In fact, η maps Ω onto $\tilde{\Omega}$. The associated operators $\mathcal{F}_D$ and $\tilde{\mathcal{F}}_D$ are defined by (23). We then have:

$$(\tilde{\mathcal{F}}_D - \mathcal{F}_D)p(z) = \int_D K_D(z,\zeta) \frac{g'(\zeta)}{\overline{g'(\zeta)}} \left(\frac{\eta'(g(\zeta))}{\overline{\eta'(g(\zeta))}} - 1 \right) \overline{p(\zeta)} dA(\zeta)$$

Multiply this by the conjugate of an arbitrary $q \in AL_2(D)$ and integrate over D, change the integrations with respect to the variables z and ζ and use the reproducing property (3) of the Bergman kernel along with $K_D(z,\zeta) = \overline{K_D(\zeta, z)}$:

$$\int_D (\tilde{\mathcal{F}}_D - \mathcal{F}_D)p\bar{q} dA = \int_D \overline{p(\zeta)} \frac{g'(\zeta)}{\overline{g'(\zeta)}} \left(\frac{\eta'(g(\zeta))}{\overline{\eta'(g(\zeta))}} - 1 \right) \left(\overline{\int_D K_D(\zeta, z) q(z) dA(z)} \right) dA(\zeta)$$

There follows

$$\Big((\tilde{\mathcal{F}}_D - \mathcal{F}_D)p, q \Big)_D = \int_D \overline{p(\zeta) q(\zeta)} \frac{g'(\zeta)}{\overline{g'(\zeta)}} \left(e^{2i \arg \eta'(g(z))} - 1 \right) dA(\zeta),$$

where $(\cdot,\cdot)_D$ on the left-hand side is the usual scalar product of $L_2(D)$ and the index D emphasizes only that the domain is the unit disc. We estimate the right-hand side by the Cauchy-Schwarz inequality and obtain

$$\left| \Big((\tilde{\mathcal{F}}_D - \mathcal{F}_D)p, q \Big)_D \right| \leq 2 \sup_{z \in D} |\sin(\arg \eta'(g(z))| \cdot \|p\|_D \cdot \|q\|_D.$$

Set $q = (\tilde{\mathcal{F}}_D - \mathcal{F}_D)p$, change the coordinates again and simplify:

$$\|(\tilde{\mathcal{F}} - \mathcal{F})f\|_\Omega \leq 2 \sup_{w \in \Omega} |\sin(\arg \eta'(w))| \cdot \|f\|_\Omega,$$

where $f = \frac{p \circ g^{-1}}{g' \circ g^{-1}}$ is the norm equivalent element in $AL_2(\Omega)$ of $p \in AL_2(D)$. Therefore we have in the operator norm

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \sup_{w \in \Omega} |\sin(\arg \eta'(w))|. \tag{37}$$

In this way, we have obtained the following

Theorem 2.22 *Let η be the conformal map of the domain Ω onto $\tilde{\Omega}$. For the Friedrichs operators of these domains there holds* (37). $\square$

Using Theorem 2.5 we additionally have

Corollary 2.23 *Let η be the conformal map of the domain Ω onto $\tilde{\Omega}$, which are both domains satisfying the conditions of Theorem 2.5. We then have*

$$\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \sup_{w \in \Omega} |\sin(\arg \eta'(w))|. \tag{38}$$

Remark 2.24 The angle between the real axis and the unit tangent vector $\tau(w)$ to the conformal image of the circle $C_\rho = \{z \in D : |z| = \rho\}$ at the point $w = g(z)$ is $\arg(izg'(z))$.

$$\arg \tilde{g}'(z) - \arg g'(z) = \arg \frac{\tilde{g}'(z)}{g'(z)} = \arg \frac{iz\tilde{g}'(z)}{izg'(z)} = \arg \tilde{\tau}(w) - \arg \tau(w)$$

So there follows

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \sup_{w \in \Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|,$$

and equivalently

$$\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \sup_{w \in \Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|.$$

If, further, the derivative of η is continuous up to the boundary (that is if $\partial\Omega$ has a continuous tangent at every point) and $\eta' \neq 0$ for the mapping in the closure of Ω, then we can restrict the supremum in Theorem 2.22 to the boundary of Ω and we obtain

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \max_{w \in \partial\Omega} |\sin(\arg \eta'(w))|, \tag{39}$$

and

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq 2 \max_{w \in \partial\Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|. \tag{40}$$

We have similarly for appropriate domains

$$\|\tilde{\mathcal{S}} - \mathcal{S}\| \leq \max_{w \in \partial\Omega} |\sin(\arg \tilde{\tau}(w) - \arg \tau(w))|. \tag{41}$$

So among domains with continuous tangent to the boundary the operator norm of the Friedrichs operator depends continuously on the shape of the domain. For domains with the additional property that Theorem 2.5 is valid on them, we have the same continuous dependence of the Schur complement operator on the domain. (For results on the domain dependence of the eigenvalues of the operators see the next sections 2.7 and 2.8.) $\square$

There is also a continuous dependence of the entries of the matrix representant $\mathcal{M}$ on the domain.

Lemma 2.25 *Let g and $\tilde{g}$ be the conformal maps of D onto Ω and $\tilde{\Omega}$, respectively. Then we have for $k = 0, 1, 2, \ldots$*

$$|\tilde{s}_k - s_k| \leq \frac{4}{k+2} \sup_{z \in D} |\sin(\arg \tilde{g}'(z) - \arg g'(z))|. \tag{42}$$

Proof. Introduce the set $D_\rho = \{z : |z| < \rho\}$, and put

$$s_k(\rho) = \frac{1}{\pi} \int_{D_\rho} \bar{z}^k \frac{\overline{g'(z)}}{g'(z)} dA(z), \quad \tilde{s}_k(\rho) = \frac{1}{\pi} \int_{D_\rho} \bar{z}^k \frac{\overline{\tilde{g}'(z)}}{\tilde{g}'(z)} dA(z).$$

Compute the difference using $\tilde{g}'(z) = \eta'(g(z)) \cdot g'(z)$

$$\tilde{s}_k(\rho) - s_k(\rho) = \frac{1}{\pi} \int_{D_\rho} \bar{z}^k \frac{\overline{g'(z)}}{g'(z)} \left(\frac{\overline{\eta'(g(z))}}{\eta'(g(z))} - 1 \right) dA(z).$$

Estimate this by taking the absolute value and using $e^{2i\alpha} - 1 = 2ie^{i\alpha} \sin \alpha$:

$$|\tilde{s}_k(\rho) - s_k(\rho)| \leq \frac{4\rho^{k+2}}{k+2} \sup_{z \in D_\rho} |\sin(\arg \tilde{g}'(z) - \arg g'(z))|,$$

since $\arg \eta'(g(z)) = \arg \tilde{g}'(z) - \arg g'(z)$. Now we get (42) by taking the limit for $\rho \to 1$. $\square$

Remark 2.26 Lemma 2.25 shows the continuous dependence of the entries of matrix $\mathcal{M}$ on the domain. Moreover, if we define a norm for the matrix $\mathcal{M}$ by

$$\|\mathcal{M}\|_* = \sup_{j,\ell} \left| \frac{j+\ell+2}{2(j+\ell)+2} [\mathcal{M}]_{j,\ell} \right| = \max_{k \geq 0} \left| \frac{k+2}{4} s_k \right|,$$

then by Lemma 2.25 we have in this norm

$$\|\tilde{\mathcal{M}} - \mathcal{M}\|_* \leq \sup_{z \in D} |\sin(\arg \tilde{g}'(z) - \arg g'(z))|, \tag{43}$$

Of course we can take the norm of the space $\ell_{(2,-1)}$ on which $\mathcal{M}$ acts and by a similar calculation as in Theorem 2.22 we can obtain the continuous dependence of $\mathcal{M}$ in the induced matrix norm instead of $\|\cdot\|_*$. $\square$

The use of the conformal mapping reveals also additional connections between the matrix representant $\mathcal{M}$ of the operators and the domain.

Lemma 2.27 *Let be $\rho \geq 1$ integer. For the quantities* (35) *we have $s_k = 0$ for $k \geq \rho$ if and only if g is a polynomial of order ρ. In this case $\mathcal{M} \in \mathbb{C}^{\rho\times\rho}$ is of rank ρ and $s_{\rho-1} = \frac{a_\rho}{\bar{a}_1} \neq 0$.*

Proof. By (36) the assumption $s_k = 0$ for $k \geq \rho$ implies

$$\int_D \bar{z}^{\rho+\ell}\frac{g'(z)}{\overline{g'(z)}}dA(z) = 0$$

for $\ell = 0, 1, 2, \ldots$. Combining these equalities with arbitrary complex numbers $\{\bar{h}_\ell\}_{\ell=0}^{\infty}$, $\sum_{\ell=0}^{\infty}|h_\ell|^2 < \infty$, we have

$$\int_D \bar{z}^{\rho}\overline{h(z)}\frac{g'(z)}{\overline{g'(z)}}dA(z) = 0,$$

where $h(z) = \sum_{\ell=0}^{\infty} h_\ell z^\ell$. Now we set $h(z) = z^\ell g'(z)$ for $\ell = 0, 1, 2, \ldots$. There follows

$$\int_D \bar{z}^{\rho+\ell}g'(z)dA(z) = 0.$$

From (29) this is equivalent to $a_{\rho+\ell+1} = 0$. The coefficients of (29) vanish for $m > \rho$, g is a polynomial of order ρ. The converse statement follows from the definition (35). Further let be $g(z) = a_0 + a_1 z + \cdots + a_\rho z^\rho$. From (35) follows also $s_{\rho-1} = \frac{a_\rho}{\bar{a}_1} \neq 0$. Because of its structure (34), $\mathcal{M}$ is therefore of rank ρ. □

Remark 2.28 If the conformal mapping function g is a polynomial of degree ρ then the image of the unit circle is a lemniscate of degree ρ, that is the locus of those points whose distances from ρ given points (the foci of the lemniscate) form a constant product. We deal with these special case in subsection 2.8, where some examples are also given. □

We can generalize the statement of the previous lemma and characterize those mappings for which the matrix $\mathcal{M}$ is infinite but of finite rank.

Lemma 2.29 *Let g be given by* (29). *The matrix $\mathcal{M}$ is of finite rank iff g is a fractional rational transformation.*

Proof. Suppose first that $\text{rank}(\mathcal{M}) = \rho + 1$ for some integer $\rho \geq 0$. From the structure of $\mathcal{M}$ there follows that the quantities (35) fulfil a certain recursion in the form

$$s_{\rho+\ell+1} = \sum_{n=0}^{\rho} \bar{\alpha}_n s_{\ell+n} \tag{44}$$

for all $\ell = 0, 1, 2, \dots$, where $\{\alpha_n\}_{n=0}^{\rho}$ are fixed complex constants. Substitute the integrals (36) into (44). There follows

$$\int_D \bar{z}^{\rho+\ell+1} \frac{\overline{g'(z)}}{g'(z)} dA(z) = \int_D \bar{z}^{\ell} \overline{\alpha(z)} \frac{\overline{g'(z)}}{g'(z)} dA(z)$$

for all $\ell = 0, 1, 2, \dots$, where $\alpha(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_\rho z^\rho$. We combine these equalities again like in Lemma 2.27 and get

$$\int_D \bar{z}^{\rho+1} \overline{h(z)} \frac{\overline{g'(z)}}{g'(z)} dA(z) = \int_D \overline{\alpha(z) h(z)} \frac{\overline{g'(z)}}{g'(z)} dA(z), \tag{45}$$

where $h(z) = h_0 + h_1 z + \dots$ is an arbitrary function. Setting $h(z) = z^\ell g'(z)$ for $\ell = 0, 1, 2, \dots$ implies

$$\int_D \bar{z}^{\rho+\ell+1} \overline{g'(z)} dA(z) = \int_D \bar{z}^{\ell} \overline{\alpha(z)} \overline{g'(z)} dA(z),$$

or, equivalently to this,

$$a_{\rho+\ell+2} = \sum_{n=0}^{\rho} \bar{\alpha}_n a_{n+\ell+1}. \tag{46}$$

This is a recursion similar to (44). Now we multiply (46) by $z^{\rho+\ell+2}$ and sum up over $\ell = 0, 1, 2, \dots$. Using (29) results into

$$g(z) - \sum_{n=0}^{\rho+1} a_n z^n = \sum_{n=0}^{\rho} \bar{\alpha}_n z^{\rho+1-n} \left[g(z) - \sum_{m=0}^{n} a_m z^m \right].$$

Solving this equation for $g(z)$ gives a fractional rational function of the form

$$g(z) = a_0 + \frac{\sum_{n=1}^{\rho+1} \left[a_n - \sum_{m=0}^{n-1} a_m \bar{\alpha}_{\rho+1-n+m} \right] z^n}{1 - \sum_{n=1}^{\rho+1} \bar{\alpha}_{\rho+1-n} z^n}. \tag{47}$$

The proof of the converse statement is merely the same calculation into the opposite direction. □

Remark 2.30 The matrix $\mathcal{M}$ and so the examined operators $\mathcal{F}$ and $\mathcal{S}$ are of finite rank for a special class of domains described in Lemma 2.27 and Lemma 2.29. Such domains are called (classical) quadrature domains and there is extensive research on such domains, see [2], [39] [40] and [44] and the references given there. A bounded domain in the complex plane is called a (classical) quadrature domain if there exist finitely many points $z_1, z_2, \ldots, z_m \in \Omega$ and coefficients $c_{kj} \in \mathbb{C}$ independently of f so that

$$\int_\Omega f dA = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} c_{kj} f^{(j)}(z_k) \tag{48}$$

for all integrable analytic functions f in Ω. The identity (48) is then called a quadrature identity and the integer $n = \sum_{k=1}^m n_k$ is the order of the quadrature identity (assuming $c_{k,n_k-1} \neq 0$). The boundary of a simply connected quadrature domain is an algebraic curve and this allows the use of the proved results in this section. □

Remark 2.31 The result of Lemma 2.29 is already known for arbitrary quadrature domains, see e.g. [39], where it is proved without the use of conformal mapping. Its use in the proof, however, reveals not only the dimension of the kernel for simply connected domains but also its structure. Namely, a reformulation of (45) implies

$$\int_D \overline{(z^{\rho+1} - \alpha(z))} \cdot \overline{h(z)} \frac{g'(z)}{\overline{g'(z)}} dA(z) = 0$$

for an arbitrary $h \in AL_2(D)$. This reveals in connection with (23) that

$$z^\ell (z^{\rho+1} - \alpha(z)) \in \ker \mathcal{F}_D \tag{49}$$

for $\ell = 0, 1, 2, \ldots$, that is the kernel of $\mathcal{F}_D$ has codimension $\rho+1$. This gives

$$\operatorname{codim}(\ker \mathcal{F}) = \operatorname{codim}(\ker(\mathcal{I} - 2\mathcal{S})) = \rho + 1$$

using the unitary correspondence between $\mathcal{F}$ and $\mathcal{F}_D$ and Theorem 2.5. In case the mapping g is of the form (47) then there exists a unique function S, called Schwarz function, holomorphic in a neighbourhood of the boundary $\partial\Omega$ with $S(w) = \bar{w}$ for $w \in \partial\Omega$, see [44]. Especially we have by setting $w = g(z)$

$$(S \circ g)(z) = \overline{g\left(\frac{1}{\bar{z}}\right)}.$$

Substituting $\frac{1}{\bar{z}}$ instead of z into (47) implies

$$(S \circ g)(z) = \bar{a}_0 + \frac{\sum_{k=0}^{\rho} \left[\bar{a}_{\rho+1-k} - \sum_{m=0}^{\rho-k} \bar{a}_m \alpha_{k+m}\right] z^k}{z^{\rho+1} - \alpha(z)}.$$

The Schwarz function of the domain transformed into polar coordinates is again a rational function and its denominator spans the kernel of the investigated operators in the sense of (49). So we have achieved also a full description of $\ker \mathcal{F}$ and $\ker(\mathcal{I} - 2\mathcal{S})$ for simply connected quadrature domains of arbitrary order. □

Theorem 2.32 *Let Ω be a simply connected quadrature domain of order $\rho + 1$, $\rho \geq 0$. Then the kernel of the associated operators $\mathcal{F}$ and $\mathcal{I} - 2\mathcal{S}$ has codimension $\rho + 1$ and is spanned by the denominator of the Schwarz function of the domain in the sense of* (49). □

The next lemma shows a connection between the area of the domain and the entries of the matrix $\mathcal{M}$.

Lemma 2.33 *Let g map the unit disc onto a domain of finite area. Then at least one of the quantities* (35) *is not zero for $k = 0, 1, \ldots$.*

Proof. Suppose conversely that

$$s_k = \frac{1}{\pi} \int_D \bar{z}^k \frac{\overline{g'(z)}}{g'(z)} dA(z) = 0$$

for all $k = 0, 1, \ldots$. Similarly to the proof of Lemma 2.27 we have

$$\int_D \overline{h(z)} \frac{\overline{g'(z)}}{g'(z)} dA(z) = 0$$

for an arbitrary function $h(z) \in AL_2(D)$. Because $g(D)$ has finite area, we have $g' \in AL_2(D)$ and we can substitute $h(z) = z^\ell g'(z)$ for an arbitrary integer $\ell \geq 0$. There follows

$$\int_D \bar{z}^\ell \overline{g'(z)} dA(z) = 0$$

and this implies $a_{\ell+1} = 0$. That is $g'(z) = 0$ which is not possible. □

Remark 2.34 If the domain is not of finite area then Lemma 2.33 is not true, see for example the mapping of the unit disc onto the halfplane or onto the exterior of a parabola (see [53]), where all the quantities (35) are zero. More generally, the matrix $\mathcal{M}$ and also the Friedrichs operator is zero (so even simpler as in case of a disc) for the so called null quadrature domains. This class consists of half-planes, exteriors of ellipses, exteriors of parabolas and some degenerate cases, see [42]. For such domains we have $\mathcal{S} = \frac{1}{2}\mathcal{I}$. (That is so, because in general the Friedrichs operator preserves constant functions on domains with finite area but for an infinite Ω the constant functions are not in $L_2(\Omega)$.) □

2.6 Comparison of the spectra

In this section we study the spectra of the investigated operators. We look for possibilities how the eigenvalues of them can be determined or estimated. Particularly important is the inf-sup constant $\beta_0(\Omega)$, the square of which is the least positive eigenvalue (10) of the Schur complement operator (see also (13)). We focus, however, rather on estimations of its value than on its exact determination since the latter is possible only in a few special cases depending on the domain. Particularly complicated is the case of domains with corners because for such domains the operators are not compact and have, besides a point spectrum, also a continuous spectrum, see e.g. [21], [14] and [15].

We demonstrate next how the eigenvalues and eigenfunctions of the various studied operators correspond to each other. First, the focus is on $\mathcal{F}_D$ defined by (23). Let $p \in AL_2(D)$ be an eigenfunction of $\mathcal{F}_D$ with the corresponding eigenvalue $\mu \in \mathbb{C}$, that is,

$$\int_D K_D(z,\zeta)\frac{g'(\zeta)}{\overline{g'(\zeta)}}\overline{p(\zeta)}dA(\zeta) = \mu p(z).$$

We see immediatly that g' is an eigenfunction to the eigenvalue 1. The same eigenvalue problem is in a variational formulation: find $p \in AL_2(D)$ and $\mu \in \mathbb{C}$ such that

$$\int_D \overline{p(\zeta)q(\zeta)}\frac{g'(\zeta)}{\overline{g'(\zeta)}}dA(\zeta) = \mu \int_D p(\zeta)\overline{q(\zeta)}dA(\zeta), \text{ for all } q \in AL_2(D).$$

By substituting $q := p$ and estimating the integral by absolute value, one easily obtains

$$|\mu| \leq 1.$$

In view of the Friedrichs inequality (18) we have more (for bounded domains): 1 is a simple eigenvalue with the corresponding eigenfunction g', the absolute value of every other eigenvalue of $\mathcal{F}_D$ is less than 1 and their corresponding eigenfunctions are orthogonal to g' with respect to the scalar product of $AL_2(D)$. (For unbounded domains $g' \notin AL_2(D)$, hence it is not an eigenfunction.)

By unitary correspondence, if μ is an eigenvalue of $\mathcal{F}_D$ with the corresponding eigenfunction $p \in AL_2(D)$, then μ is also an eigenvalue of $\mathcal{F}$ with the corresponding eigenfunction

$$f = \frac{p \circ g^{-1}}{g' \circ g^{-1}} \in AL_2(\Omega).$$

Hence $\mathcal{F}$ and $\mathcal{F}_D$ have the same spectrum. The eigenfunction of $\mathcal{F}$ corresponding to the simple eigenvalue 1 is now the function identically 1 in the bounded domain Ω. The eigenfunctions of the other eigenvalues are orthogonal to this function, i.e. their integral over the domain Ω is zero.

If a matrix representation $\mathcal{M}$ of $\mathcal{F}_D$ is possible, then the vector $p \in \ell_{(2,-1)}$ composed of the coefficients of the Taylor series expansion of the eigenfunction $p \in AL_2(D)$ fulfils

$$\mathcal{M}\mathcal{C}p = \mu p,$$

whence it is an eigenvector corresponding to the eigenvalue μ, see section 2.4.

If we wish to relate the spectrum of $\mathcal{S}$ to the spectra of the operators $\mathcal{F}$, $\mathcal{F}_D$ and $\mathcal{M}\mathcal{C}$, then we observe first that they are conjugate-linear while $\mathcal{S}$ is complex-linear. If $f \in AL_2(\Omega)$ is an eigenfunction to the eigenvalue $\mu \in \mathbb{C}$:

$$\mathcal{F}f = \mu f,$$

then by (15) there follows

$$\mathcal{F}\left(e^{\frac{1}{2}i\arg\mu}f\right) = |\mu|e^{\frac{1}{2}i\arg\mu}f,$$

which implies using (20) and (21)

$$\begin{aligned}
\mathcal{S}p_R &= \frac{1}{2}p_R - \operatorname{Re}\left(|\mu|e^{\frac{1}{2}i\arg\mu}f\right) = \frac{1-|\mu|}{2}p_R,\\
\mathcal{S}p_I &= \frac{1}{2}p_I + \operatorname{Im}\left(|\mu|e^{\frac{1}{2}i\arg\mu}f\right) = \frac{1+|\mu|}{2}p_I,
\end{aligned}$$

where we have set now $p_R = 2\operatorname{Re}(e^{\frac{1}{2}i\arg\mu}f)$ and $p_I = 2\operatorname{Im}(e^{\frac{1}{2}i\arg\mu}f)$. This means that one eigenpair (μ, f) of $\mathcal{F}$ with a complex eigenvalue $|\mu| < 1$ gives rise to two eigenpairs of $\mathcal{S}$, where the eigenvalues

$$\lambda_\pm := \frac{1 \pm |\mu|}{2}$$

lie symmetric with respect to $\frac{1}{2}$ and the corresponding (real valued harmonic pressure) eigenfunctions are the real and imaginary part of the eigenfunction of $\mathcal{F}$ multiplied by the factor $e^{\frac{1}{2}i\arg\mu}$. Different is the case of the simple eigenvalue 1, because in this case $f \equiv 1$ in Ω and also $\mathcal{F}1 \equiv 1$. Hence $p_R = 2\operatorname{Re} f \equiv 2$ (i.e. constant pressure) is an eigenfunction to the eigenvalue 0 but $p_I = 2\operatorname{Im} f \equiv 0$ is not an eigenfunction at all. In fact, this result is the same as what we can see from the Crouzeix-Velte decomposition (12) along with Remark 2.7.

Let us summarize the results:

Theorem 2.35 *The spectra of the operators $\mathcal{F}$ and $\mathcal{F}_D$ are identical, their eigenfunctions to the same eigenvalue being unitarily equivalent.*

If the infinite matrix $\mathcal{M}$ defines a bounded operator on $\ell_{(2,-1)}$, then the spectrum of the conjugate-linear operator $\mathcal{MC}$ is identical with the spectra of $\mathcal{F}$ and $\mathcal{F}_D$.

Every eigenvalue $\mu \neq 1$ of the operator $\mathcal{F}$ to the eigenfunction f gives rise to two eigenvalues $\lambda_\pm$ of the Schur complement operator $\mathcal{S}$. These eigenvalues are $\lambda_\pm = \frac{1 \pm |\mu|}{2} \in (0,1)$ with the eigenfunctions $p_R = 2\,Re(e^{\frac{1}{2} i \arg \mu} f)$ and $p_I = 2\,Im(e^{\frac{1}{2} i \arg \mu} f)$. The simple eigenvalue $\mu = 1$ of $\mathcal{F}$ gives rise to the eigenvalue $\lambda = 0$ of $\mathcal{S}$ with the constant function as eigenfunction for both operators. □

We have some conclusions from Theorem 2.35.

Remark 2.36 Denote $\mu_2 := \max\{|\mu| : \mu \in \sigma(\mathcal{F}) \setminus \{1\}\}$. As a consequence of the Friedrichs inequality (18) we have $\mu_2 = \gamma_\Omega < 1$ and by Theorem 2.35 we can compute the inf-sup constant (10) of the domain Ω by

$$\beta_0^2(\Omega) = \frac{1 - \mu_2}{2} = \frac{1 - \gamma_\Omega}{2}.$$

This is an important consequence, however, it is already known, see e.g. [45], [46]. □

Remark 2.37 If Ω is a conformal map of the unit disc D by a polynomial or rational conformal mapping (that is, Ω is a simply connected quadrature domain), then $\mathcal{F}$, $\mathcal{F}_D$ and $\mathcal{MC}$ have a non trivial kernel of finite codimension, see Lemma 2.29 and the remarks thereafter. For such quadrature domains the Schur complement operator of the Stokes problem has also a finite number of eigenvalues which can be computed via the eigenvalue problem for $\mathcal{MC}$ if the conformal mapping onto the domain is already known. The simplest example of a bounded simply connected quadrature domain is the unit disc D for which it is known (see e.g. [54]) that the only eigenvalue of $\mathcal{S}$ in $(0,1)$ is $\frac{1}{2}$ which is of infinite multiplicity. (In this case the kernel of $\mathcal{F}$ has codimension 1.) For simply connected quadrature domains the eigenvalue $\lambda = \frac{1}{2}$ remains an eigenvalue of infinite multiplicity, but $\mathcal{S}$ has further a finite number of additional eigenvalues in $(0,1)$ as explained in Theorem 2.35. These additional eigenvalues – and eigenfunctions – can be computed with the help of the matrix $\mathcal{M}$. □

Remark 2.38 An additional important consequence of Theorem 2.35 is that the Crouzeix-Velte subspace investigated in [45] is not reduced to zero for domains arising as conformal maps of the unit disc by a polynomial mapping function. □

The lemmas in the previous section and Remark 2.34 imply the following.

Corollary 2.39 *The Schur complement operator $\mathcal{S}$ has the spectrum $\left\{0, \frac{1}{2}, 1\right\}$ iff the domain is a quadrature domain of order less than 2.*

The question whether the spectrum of the examined operators characterize the domain in the general case is answered in [39]. Although it is true for quadrature domains of order one (i.e. discs) and two, but it is not true in case of order three.

Proposition 2.40 *(Proposition 4.9 in [39]) There exists a continuous family of quadrature domains of order three with the same Friedrichs operator (up to unitary equivalence) and such that no two domains in the family are related by an affine transformation of $\mathbb{C}$.* □

"Affine transformation" means here a function $w \mapsto aw + b$ for some $a, b \in \mathbb{C}$. Such transformations are reflections, rotations, translations and similarities. It is obvious that the operators of domains, which are affine maps of each other, are unitarily equivalent.

Remark 2.41 So the inf-sup constant also does not characterize the domain, even not up to affine transformations of the plane. □

Remark 2.42 If the conformal mapping g has real coefficients then the domain Ω is symmetric to the real axis and $\mathcal{M}$ has also real entries. In this case the eigenproblem for $\mathcal{MC}$ is equivalent to

$$\mathcal{D}^{-\frac{1}{2}}\mathcal{M}\mathcal{D}^{\frac{1}{2}}w = -2\mu w$$

where $w = \mathcal{D}^{-\frac{1}{2}}p$ with the real symmetric matrix $\mathcal{D}^{-\frac{1}{2}}\mathcal{M}\mathcal{D}^{\frac{1}{2}}$. So all eigenvalues of $\mathcal{M}$ are real. (If we calculate these eigenvalues numerically it is worth to transform $\mathcal{M}$ to this form.) □

The preceding remark suggest a correspondence between the symmetric geometric properties of the domain and the spectrum of the studied operators. We examine this in the following

Theorem 2.43 *Let the bijective conformal mapping of the unit disc be of the form*

$$g(z) = \sum_{n=0}^{\infty} a_{nM+1} z^{nM+1}, \tag{50}$$

where $a_1 \neq 0$ and $M \geq 2$ is an arbitrary integer, then there are eigenvalues of the Schur complement operator with multiplicity more than 1.

Proof. If g is of the form (50) then we have, by multiplying the derivative of (50) and (30),

$$1 = g'(z) \cdot \frac{1}{g'(z)} = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{M-1} \left[\sum_{\ell=0}^{n} (\ell M + 1) a_{\ell M+1} b_{(n-\ell)M+k} \right] z^{nM+k} \right\}.$$

This can be written in the infinite block matrix form

$$\begin{pmatrix} a_1 I & 0 & \dots & & & & \\ (M+1)a_{M+1} I & a_1 I & 0 & \dots & & & \\ \ddots & \ddots & \ddots & \ddots & & & \\ (nM+1)a_{nM+1} I & \dots & \dots & a_1 I & 0 & \dots & \\ \ddots & & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix},$$

where $I \in \mathbb{R}^{M \times M}$ is the identity matrix, $\vec{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^M$ and

$$\tilde{b}_n := (b_{nM}, b_{nM+1}, \dots, b_{nM+M-1})^T$$

for $n = 0, 1, \dots$. There follows $\tilde{b}_0 = \frac{1}{a_1} \vec{e}_1$ and further by induction $\tilde{b}_n = C \vec{e}_1$ where $C \in \mathbb{C}$ is a constant composed of the coefficients of (50). This implies that, in (30), $b_\ell \neq 0$ only for $\ell = nM$ where $n = 0, 1, \dots$. From the series form of $g(z)$ we see that $a_k \neq 0$ only for $k = nM + 1$. This means that $s_\ell \neq 0$ only for $\ell = nM$ where $n = 0, 1, \dots$.

For $k = 0, 1, \dots, [\frac{M}{2}]$, let us introduce the pairwise orthogonal subspaces L_k of $\ell_{(2,-1)}$ as follows. We shall say that $v := (v_0, v_1, \dots)^T \in L_k$, if $v \in \ell_{(2,-1)}$ and if

$$v_\ell = 0 \text{ for all } \ell \neq nM + k, (n+1)M - k,$$

where $n = 0, 1, \dots$ and k is a fixed integer between 0 and $[\frac{M}{2}]$. If $v \in L_k$ for $k \neq 0, \frac{M}{2}$ ($\frac{M}{2}$ only in case M is even) is an eigenvector of $\mathcal{MC}$ to the eigenvalue μ then $\tilde{v}$ is an eigenvector to the eigenvalue $-\mu$ where the elements of $\tilde{v}$ are defined by

$$\begin{aligned} \tilde{v}_{nM+k} &= v_{nM+k}, \\ \tilde{v}_{(n+1)M-k} &= -v_{(n+1)M-k} \end{aligned}$$

for $n = 0, 1, \dots$, otherwise $\tilde{v}_\ell = v_\ell = 0$. Therefore $\frac{1+\mu}{2}$ and $\frac{1-\mu}{2}$ are eigenvalues of the Schur complement with twice the multiplicity of μ. □

Remark 2.44 If the mapping function is of the form as in Theorem 2.43, then

$$g\left(e^{ik\frac{2\pi}{M}}z\right) = e^{ik\frac{2\pi}{M}}g(z), \text{ where } k = 0, 1, \ldots, M-1. \tag{51}$$

The left-hand side of this equality means that we first rotate the unit disc by $k\frac{2\pi}{M}$ and then we map it onto the domain Ω. The right-hand side means that we first map the unit disc onto Ω and then we rotate Ω by $k\frac{2\pi}{M}$. Therefore Ω is invariant under any rotation by the angle $k\frac{2\pi}{M}$, consequently Ω has M symmetry axes. Theorem 2.43 says that in the case of symmetrical domains the Schur complement operator has multiply eigenvalues depending on the number of the symmetry axes (so the inf-sup constant of the domain could be also a multiple eigenvalue). Moreover each subspace L_k of $\ell_{(2,-1)}$ seems to be connected with one or two symmetry axes. □

Remark 2.45 The multiplicity of the eigenvalues of the studied operators of symmetric domains can be investigated also without using the matrix representation. Let g map the unit disc conformally onto the domain Ω so that $g(0) = 0$. Fix an angle $0 < \theta < 2\pi$ and define

$$g_\theta(z) := e^{-i\theta}g(e^{i\theta}z). \tag{52}$$

The function $g_\theta(z)$ maps the unit disc onto a domain Ω_θ, which is the clockwise rotated image of Ω around the origin by the angle θ. For the corresponding Friedrichs operators we have by calculation using (23)

$$(\mathcal{F}_D^\theta p_\theta)(z) = e^{i\theta}(\mathcal{F}_D p)(e^{i\theta}z) \tag{53}$$

for a function $p \in AL_2(D)$. If the function p is an eigenfunction of $\mathcal{F}_D$ to the eigenvalue μ, i.e. $\mathcal{F}_D p = \mu p$, then there follows by (53)

$$(\mathcal{F}_D^\theta p_\theta)(z) = e^{i\theta}\mu p(e^{i\theta}z) = e^{2i\theta}\mu p_\theta(z),$$

which means that the function p_θ is an eigenfunction of $\mathcal{F}_D^\theta$ to the eigenvalue $e^{2i\theta}\mu$. Now, if $g_\theta(z) = g(z)$ for $z \in D$, then we have $\Omega_\theta = \Omega$ (the domain Ω has a rotational symmetry with the angle θ) and by (23) we also obtain $\mathcal{F}_D^\theta = \mathcal{F}_D$. This implies that, $e^{2i\theta}\mu$ and p_θ is an eigenpair of $\mathcal{F}_D$ along with the eigenpair μ and p. By Theorem 2.35 the Schur complement operator of the domain Ω has multiple eigenvalues because $|\mu| = |e^{2i\theta}\mu|$. □

2.7 Estimation of the constants

The knowledge of the exact value of the inf-sup constant of a domain is very useful (see e.g. [46]). However, its determination is not an easy task. In most cases one must be satisfied with some estimations of the inf-sup constant. Particularly important is the estimation from below. Such an estimation is derived for example in [46] for star-shaped domains using a result from [26].

We show in this section how conformal mapping can be useful to obtain such estimates. We start with an alternative to the Friedrichs inequality (18): there exists a constant $\Gamma_\Omega \geq 1$ depending only on the shape of the domain Ω such that

$$\int_\Omega u^2 dA \leq \Gamma_\Omega \int_\Omega v^2 dA \tag{54}$$

for all conjugate harmonic functions u and v in $L_2(\Omega)$ satisfying

$$\int_\Omega u dA = 0, \tag{55}$$

see [21]. There is another possibility for the normalization of the harmonic function u which leads to another inequality:

$$\int_\Omega u^2 dA \leq \tilde{\Gamma}_\Omega \int_\Omega v^2 dA \tag{56}$$

for all conjugate harmonic functions u and v in $L_2(\Omega)$ satisfying

$$u(w_0) = 0, \tag{57}$$

for some $w_0 \in \Omega$, moreover we have $\Gamma_\Omega \leq \tilde{\Gamma}_\Omega$ see [26]. (For the disc and $w_0 = 0$ the normalizations (55) and (57) are the same by the mean-value theorem.)

Using this along with Remark 2.36 there follows

$$\gamma_\Omega = \frac{\Gamma_\Omega - 1}{\Gamma_\Omega + 1}, \tag{58}$$

$$\beta_0^2(\Omega) = \frac{1}{\Gamma_\Omega + 1}, \tag{59}$$

which implies also

$$\beta_0^2(\Omega) \leq \frac{1}{2}, \tag{60}$$

where the equality holds for example if $\Omega = D$.

If $\Omega = g(D)$ and u, v are conjugate harmonic functions in D, then $u \circ (g^{-1})$ and $v \circ (g^{-1})$ are conjugate harmonic functions in Ω. So we have as in [27]:

$$\Gamma_\Omega \leq \frac{\int_D u^2 |g'|^2 dA}{\int_D v^2 |g'|^2 dA} \leq \frac{\sup_D |g'|^2 \int_D u^2 dA}{\inf_D |g'|^2 \int_D v^2 dA} = \frac{\sup_D |g'|^2}{\inf_D |g'|^2} \tag{61}$$

because $\Gamma_D = 1$ (and so $\int_D u^2 dA = \int_D v^2 dA$ if $u(0) = \int_D u dA = 0$). Using also the maximum principle (see [37]) we obtain

$$\Gamma_\Omega \leq \left(\frac{\sup_{\partial D} |g'|}{\inf_{\partial D} |g'|} \right)^2 \tag{62}$$

However this estimation is usable only if $\partial\Omega$ has a continuous tangent and hence Ω does not have any corners (in the presence of corners $\sup_{\partial D} |g'|^2 = \infty$) and if Ω does not have any internal cusps (in case of internal cusps on $\partial\Omega$, $\inf_{\partial D} |g'|^2 = 0$).

Corollary 2.46 *Let $\Omega = g(D)$ be such that the derivative g' of the conformal map is continuous on the closure of D and $|g'|$ has a positive lower and upper bound on ∂D. Then we have*

$$\frac{1}{\sqrt{2}} \cdot \frac{\inf_{z\in\partial D} |g'(z)|}{\sup_{z\in\partial D} |g'(z)|} \leq \beta_0(\Omega) \leq \frac{1}{\sqrt{2}} \tag{63}$$

for the inf-sup constant (10) *of the domain Ω. The equalities hold for $g(z) = z$, that is, $\Omega = D$.*

Proof. By (60) and (59) there follows

$$\frac{1}{2} \geq \beta_0^2(\Omega) \geq \frac{1}{1 + \left(\frac{\sup_{\partial D} |g'|}{\inf_{\partial D} |g'|}\right)^2} \geq \frac{1}{2} \left(\frac{\inf_{\partial D} |g'|}{\sup_{\partial D} |g'|} \right)^2 .$$

If $g(z) = z$, then $g'(z) \equiv 1$, so the equations are valid in the preceding estimation. This completes the proof. □

Remark 2.47 Instead of the conformal mapping g of the unit disc onto the domain Ω we can use in the preceding Corollary 2.46 also its inverse, the Riemann map $R(w) := g^{-1}(w)$ of Ω onto the unit disc:

$$\frac{1}{\sqrt{2}} \cdot \frac{\inf_{w\in\partial\Omega} |R'(w)|}{\sup_{w\in\partial\Omega} |R'(w)|} \leq \beta_0(\Omega) \leq \frac{1}{\sqrt{2}} \tag{64}$$

If, moreover, the boundary of Ω is C^2 smooth and $R(a) = 0$, $R'(a) > 0$ for a fixed $a \in \Omega$, then the Riemann map fulfils

$$T(w)R'(w) + \int_{\omega\in\partial\Omega} N(w,\omega)T(\omega)R'(\omega)|d\omega| = -\frac{\overline{T(w)}}{R'(a)(\bar{w} - \bar{a})^2}$$

for $w \in \partial\Omega$, see e.g. [41]. The kernel N is defined by

$$N(w,\omega) = \begin{cases} \frac{1}{\pi}\operatorname{Im}\frac{T(w)}{w-\omega}, & w,\omega \in \partial\Omega, w \neq \omega, \\ \frac{1}{2\pi}\operatorname{Im}\frac{w''(t)\overline{w'(t)}}{|w'(t)|^3}, & w,\omega \in \partial\Omega, w = \omega, \end{cases}$$

where $w = w(t)$ is a parametrization of $\partial\Omega$ and $T(w) = \frac{w'(t)}{|w'(t)|}$ is the unit tangent to the boundary at $w \in \partial\Omega$. This boundary integral equation can also be used to evaluate the estimation (64) numerically. □

Remark 2.48 By a theorem of Kellogg [28], if Ω is a domain bounded by a smooth closed Jordan curve for which the angle of inclination $\phi(s)$ of the tangent to the real axis, as a function of the arc length s of $\partial\Omega$, satisfies a Hölder condition

$$|\phi(s_1) - \phi(s_2)| \leq k|s_1 - s_2|^\alpha$$

with some $k > 0$ and $0 < \alpha < 1$ (Lyapunov curve), then $g' \neq 0$ on $\bar{D} := D \cup \partial D$ and the function $\log|g'|$ satisfies also a Hölder condition on ∂D of the same order, i.e.

$$\left|\log|g'(e^{i\theta_1})| - \log|g'(e^{i\theta_2})|\right| \leq K|\theta_1 - \theta_2|^\alpha,$$

for some $K > 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$. Now let us denote by $\theta_{\max}$ and $\theta_{\min}$ those values of the variable θ for which the continuous function $\theta \mapsto |g'(e^{i\theta})|$ takes its maximum and minimum, respectively. There follows

$$\log\left|\frac{g'(e^{i\theta_{\max}})}{g'(e^{i\theta_{\min}})}\right| = \left|\log|g'(e^{i\theta_{\max}})| - \log|g'(e^{i\theta_{\min}})|\right| \leq K|\theta_{\max} - \theta_{\min}|^\alpha \leq K(2\pi)^\alpha,$$

and therefore we obtain

$$\left|\frac{g'(e^{i\theta_{\max}})}{g'(e^{i\theta_{\min}})}\right| \leq \exp(K(2\pi)^\alpha).$$

This gives by Corollary 2.46 a possibility for the estimation of the inf-sup constant of such domains:

$$\beta_0(\Omega) \geq \frac{1}{\sqrt{2}}\exp(-K(2\pi)^\alpha), \tag{65}$$

where K depends (via k) on the shape of the domain. If the second derivative of the conformal map g is continuous in $\overline{\Omega}$, then we have

$$\log g'(e^{i\theta_1}) - \log g'(e^{i\theta_2}) = \int_{\theta_2}^{\theta_1} i\frac{e^{i\theta}g''(e^{i\theta})}{g'(e^{i\theta})}d\theta,$$

and by taking the real parts of both sides

$$\log|g'(e^{i\theta_1})| - \log|g'(e^{i\theta_2})| = -\int_{\theta_2}^{\theta_1} \operatorname{Im} \frac{e^{i\theta} g''(e^{i\theta})}{g'(e^{i\theta})} d\theta.$$

We estimate the integral by the Hölder inequality:

$$\begin{aligned}
\left|\log|g'(e^{i\theta_1})| - \log|g'(e^{i\theta_2})|\right| &\leq \int_{\theta_1}^{\theta_2} \left|\operatorname{Im} \frac{e^{i\theta} g''(e^{i\theta})}{g'(e^{i\theta})}\right| d\theta \\
&\leq \left(\int_{\theta_1}^{\theta_2} \left|\operatorname{Im} \frac{e^{i\theta} g''(e^{i\theta})}{g'(e^{i\theta})}\right|^{\frac{1}{1-\alpha}} d\theta\right)^{1-\alpha} \left(\int_{\theta_1}^{\theta_2} 1^{\frac{1}{\alpha}} d\theta\right)^{\alpha} \\
&\leq \left(\int_{0}^{2\pi} \left|\operatorname{Im} \frac{e^{i\theta} g''(e^{i\theta})}{g'(e^{i\theta})}\right|^{\frac{1}{1-\alpha}} d\theta\right)^{1-\alpha} |\theta_1 - \theta_2|^{\alpha}.
\end{aligned}$$

This shows that the constant K in (65) can be estimated as

$$K \leq \left(\int_{\partial D} \left|\operatorname{Im} \frac{z g''(z)}{g'(z)}\right|^{\frac{1}{1-\alpha}} |dz|\right)^{1-\alpha},$$

if the integral exists. □

Remark 2.49 Another possibility offers if we use another result of the Kellogg theorem, i.e.

$$\left|g'(e^{i\theta_1}) - g'(e^{i\theta_2})\right| \leq \tilde{K}|\theta_1 - \theta_2|^{\alpha},$$

where $\tilde{K} > 0$. We substitute $\theta_{\max}$ and $\theta_{\min}$ again and estimate by the triangle inequality:

$$\begin{aligned}
\left||g'(e^{i\theta_{\max}})| - |g'(e^{i\theta_{\min}})|\right| &\leq \left|g'(e^{i\theta_{\max}}) - g'(e^{i\theta_{\min}})\right| \\
&\leq \tilde{K}|\theta_{\max} - \theta_{\min}|^{\alpha} \leq \tilde{K}(2\pi)^{\alpha}.
\end{aligned}$$

There follows

$$\left|\left|\frac{g'(e^{i\theta_{\max}})}{g'(e^{i\theta_{\min}})}\right| - 1\right| \leq \frac{\tilde{K}(2\pi)^{\alpha}}{m},$$

where we have set $m := |g'(e^{i\theta_{\min}})| > 0$. This gives for the inf-sup constant:

$$\beta_0(\Omega) \geq \frac{1}{\sqrt{2}} \cdot \frac{m}{m + \tilde{K}(2\pi)^{\alpha}},$$

where we have for sufficiently smooth conformal map

$$\tilde{K} \leq \left(\int_{\partial D} |g''(z)|^{\frac{1}{1-\alpha}} |dz|\right)^{1-\alpha}$$

by a similar calculation as in the previous Remark 2.48. □

Remark 2.50 For another utilization of Corollary 2.46 we can use the representation

$$g'(z) = g'(0) \exp \left(\frac{1}{\pi} \int_{\partial D} \frac{\psi(\theta) d\zeta}{\zeta - z} \right), \zeta = e^{i\theta}, z \in D,$$

where $\psi(\theta) := \phi(\theta) - \theta - \frac{\pi}{2}$ and $\phi = \phi(\theta)$ denotes again the angle between the tangent to $\partial\Omega$ at $g(e^{i\theta})$ and the real axis. If g' is continuous on the closed unit disc, then

$$g'(e^{i\theta}) = g'(0) e^{\tilde{\psi}(\theta)} e^{i\psi(\theta)},$$

where $\tilde{\psi}$ is the conjugate function to ψ in the sense of the Cauchy principal value of the improper integral

$$\tilde{\psi}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot \frac{\theta - t}{2} dt.$$

If ψ satisfies a Hölder condition of order $0 < \alpha < 1$, then $\tilde{\psi}(\theta)$ exists for any θ and satisfies this condition as well. Thus we have

$$\frac{\sup_{\partial D} |g'|}{\inf_{\partial D} |g'|} = \frac{\sup_{0 \leq \theta < 2\pi} e^{\tilde{\psi}(\theta)}}{\inf_{0 \leq \theta < 2\pi} e^{\tilde{\psi}(\theta)}},$$

which can be substituted into Corollary 2.46 to obtain an estimation for the inf-sup constant. In this estimation the function describing the angle of the tangent to $\partial\Omega$ is involved indirectly via its conjugate function. □

We can generalize calculation (61) which has led to the estimation (62).

Theorem 2.51 *Let the domains Ω and $\tilde{\Omega}$ have smooth boundaries such that the derivative of the bijective conformal mapping η of Ω onto $\tilde{\Omega}$ is continuous on the closure of Ω and $|\eta'|$ has a positive lower and upper bound on $\partial\Omega$. Let be $0 \in \Omega$ and the conformal mapping normalized so that $\eta(0) = 0$. We have*

$$\frac{\inf_{\partial\Omega} |\eta'|^2}{\sup_{\partial\Omega} |\eta'|^2} \tilde{\Gamma}_\Omega \leq \tilde{\Gamma}_{\tilde{\Omega}} \leq \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} \tilde{\Gamma}_\Omega \tag{66}$$

for the constants $\tilde{\Gamma}_\Omega$ and $\tilde{\Gamma}_{\tilde{\Omega}}$ from the inequality (56).

Proof. First observe that $0 \in \tilde{\Omega}$. Let be $\tilde{u}$ a harmonic function in $\tilde{\Omega}$ such that $\tilde{u}(0) = 0$ holds. Then $u = \tilde{u} \circ \eta$ is harmonic in Ω and satisfies $u(0) = 0$. Let $\tilde{u}$ and $\tilde{v}$ be such that

$$\tilde{\Gamma}_{\tilde{\Omega}} = \frac{\int_{\tilde{\Omega}} \tilde{u}^2 dA}{\int_{\tilde{\Omega}} \tilde{v}^2 dA},$$

where $\tilde{u}$ is normalized by $\tilde{u}(0) = 0$. We have similar to (61)

$$\tilde{\Gamma}_{\tilde{\Omega}} = \frac{\int_{\tilde{\Omega}} \tilde{u}^2 dA}{\int_{\tilde{\Omega}} \tilde{v}^2 dA} \leq \frac{\sup_{\Omega} |\eta'|^2}{\inf_{\Omega} |\eta'|^2} \frac{\int_{\Omega} u^2 dA}{\int_{\Omega} v^2 dA} \leq \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} \tilde{\Gamma}_{\Omega},$$

which is the right-hand side of (66). The left-hand side can be similarly obtained if we calculate using the inverse of the mapping η. □

Remark 2.52 Suppose that we have an upper estimate M for $\tilde{\Gamma}_{\Omega}$, then

$$\Gamma_{\Omega} \leq \tilde{\Gamma}_{\Omega} \leq M,$$

that is, M is also an upper estimate for Γ_{Ω}. By Theorem 2.51 there follows

$$\Gamma_{\tilde{\Omega}} \leq \tilde{\Gamma}_{\tilde{\Omega}} \leq \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} M.$$

By (59) we have $\beta_0^2(\Omega) \geq \frac{1}{1+M}$ and

$$\beta_0^2(\tilde{\Omega}) \geq \frac{1}{1 + \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} M} \geq \frac{\inf_{\partial\Omega} |\eta'|^2}{\sup_{\partial\Omega} |\eta'|^2} \cdot \frac{1}{1+M}.$$

Therefore, if we have a lower estimate for the inf-sup constant of the domain Ω, then we also have a lower estimate for the other domain $\tilde{\Omega} = \eta(\Omega)$. □

A slight modification of Theorem 2.51 gives

Theorem 2.53 *Let the domains Ω and $\tilde{\Omega}$ have smooth boundaries such that the derivative of the bijective conformal mapping η of Ω onto $\tilde{\Omega}$ is continuous on the closure of Ω and $|\eta'|$ has a positive lower and upper bound on $\partial\Omega$. Then we have*

$$\Gamma_{\tilde{\Omega}} \leq \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} \Gamma_{\Omega}. \tag{67}$$

Proof. Set $\tilde{w} = \eta(w)$. Let $\tilde{u}$ be a harmonic function on $\tilde{\Omega}$ such that for the harmonic function $u := \tilde{u} \circ \eta$ on Ω we have

$$0 = \int_{\Omega} u(w) dA(w) = \int_{\tilde{\Omega}} \tilde{u}(\tilde{w}) \left| \frac{1}{\eta'(\eta^{-1}(\tilde{w}))} \right|^2 dA(\tilde{w}). \tag{68}$$

That is, the integral of the harmonic function $\tilde{u}$ with the positive weight $|\eta'(\eta^{-1}(\tilde{w})|^{-2}$ over the domain $\tilde{\Omega}$ is zero. Hence a $\tilde{w}_0 \in \tilde{\Omega}$ must exist for which $\tilde{u}(\tilde{w}_0) = 0$. Choosing this $\tilde{w}_0$ according to (57) gives

$$\tilde{\Gamma}_{\tilde{\Omega}} = \frac{\int_{\tilde{\Omega}} \tilde{u}^2 dA}{\int_{\tilde{\Omega}} \tilde{v}^2 dA}$$

with conjugate harmonic functions $\tilde{u}$ and $\tilde{v}$. There follows by (68) and (54), (55)

$$\tilde{\Gamma}_{\tilde{\Omega}} = \frac{\int_{\tilde{\Omega}} \tilde{u}^2 dA}{\int_{\tilde{\Omega}} \tilde{v}^2 dA} \leq \frac{\sup_\Omega |\eta'|^2}{\inf_\Omega |\eta'|^2} \frac{\int_\Omega u^2 dA}{\int_\Omega v^2 dA} \leq \frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} \Gamma_\Omega,$$

which completes the proof in view of $\Gamma_{\tilde{\Omega}} \leq \tilde{\Gamma}_{\tilde{\Omega}}$. □

Corollary 2.54 *Let Ω and $\tilde{\Omega}$ be as in Theorem 2.53. Then there follows*

$$\beta_0(\tilde{\Omega}) \geq \frac{\inf_{\partial\Omega} |\eta'|}{\sup_{\partial\Omega} |\eta'|} \beta_0(\Omega).$$

Proof. Use (59), the result of Theorem 2.53 and $\frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2} \geq 1$.

$$\begin{aligned} \beta_0^2(\tilde{\Omega}) &= \frac{1}{1+\Gamma_{\tilde{\Omega}}} \geq \frac{1}{1+\frac{\sup_{\partial\Omega} |\eta'|^2}{\inf_{\partial\Omega} |\eta'|^2}\Gamma_\Omega} \geq \frac{\inf_{\partial\Omega} |\eta'|^2}{\sup_{\partial\Omega} |\eta'|^2} \frac{1}{1+\Gamma_\Omega} \\ &= \left(\frac{\inf_{\partial\Omega} |\eta'|}{\sup_{\partial\Omega} |\eta'|} \right)^2 \beta_0^2(\Omega). \end{aligned}$$

□

Example 2.55 We illustrate Theorem 2.53 on an example using the mapping properties of the function $\tilde{w} = \eta(w) = e^w$, which maps [37] the rectangle

$$\Omega = \{w \in \mathbb{C} : \log r \leq \operatorname{Re} w \leq \log R, 0 \leq \operatorname{Im} w \leq \theta\},$$

where $0 < r < R$, $|\theta| < 2\pi$, conformally onto the annular sector

$$\tilde{\Omega} = \{\tilde{w} \in \mathbb{C} : r \leq |\tilde{w}| \leq R, 0 \leq \arg w \leq \theta\}.$$

We have $\eta'(w) = e^w$, which gives $|\eta'(w)|^2 = |e^w|^2 = e^{2 \operatorname{Re} w}$ and

$$r^2 \leq |\eta'(w)|^2 \leq R^2 \text{ for } w \in \Omega.$$

By Theorem 2.53 and the corollary thereafter, we obtain

$$\Gamma_{\tilde{\Omega}} \leq \Gamma_\Omega \frac{R^2}{r^2} \text{ and } \beta_0(\tilde{\Omega}) \geq \beta_0(\Omega) \frac{r}{R}. \tag{69}$$

Now we use a lower estimate for the inf-sup constant of the rectangle Ω given in [46]: $\beta_0(\Omega) \geq \frac{1}{L} \sin \frac{\pi}{8}$, where $L \geq 1$ is the ratio between the sides of the rectangle. In this case we have either

$$L = \frac{\log \frac{R}{r}}{\theta} \text{ or } L = \frac{\theta}{\log \frac{R}{r}},$$

depending on the values of the parameter r, R and θ. If we substitute one of these expressions into (69), then we obtain a lower estimation of the inf-sup constant $\beta_0(\tilde{\Omega})$ of an annular sector depending only on its geometric properties. □

The following theorem is similar to the previous results and useful in relating the inf-sup constants of two domains:

Theorem 2.56 *Let g be the bijective conformal mapping of the unit disc D onto the domain Ω so that $g(0) = g'(0) - 1 = 0$. Let the univalent function $\tilde{g}$ be such that $\tilde{g}(0) = \tilde{g}'(0) - 1 = 0$ and*

$$|\tilde{g}'(z) - g'(z)| \leq \varepsilon |g'(z)| \tag{70}$$

for all $z \in D \cup \partial D$ with some $0 < \varepsilon < 1$. Then $\tilde{g}$ maps D conformally onto another domain $\tilde{\Omega}$ and there follows

$$\frac{1-\varepsilon}{1+\varepsilon}\beta_0(\Omega) \leq \beta_0(\tilde{\Omega}) \leq \frac{1+\varepsilon}{1-\varepsilon}\beta_0(\Omega) \tag{71}$$

for the inf-sup constants of the domains.

Proof. From the conditions of the theorem we have by the triangle inequality

$$||\tilde{g}'(z)| - |g'(z)|| \leq |\tilde{g}'(z) - g'(z)| \leq \varepsilon |g'(z)|.$$

This gives

$$0 < (1-\varepsilon)|g'(z)| \leq |\tilde{g}'(z)| \leq (1+\varepsilon)|g'(z)|, \tag{72}$$

hence $\tilde{g}$ is also conformal. There also follows

$$\frac{1}{1+\varepsilon}|\tilde{g}'(z)| < |g'(z)| < \frac{1}{1-\varepsilon}|\tilde{g}'(z)|. \tag{73}$$

Using inequality (72) we estimate similar as in the proof of Theorem 2.53 and we obtain

$$\Gamma_{\tilde{\Omega}} \leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\Gamma_{\Omega}.$$

By the relations (58) and (59), there follows

$$\beta_0(\tilde{\Omega}) \geq \frac{1-\varepsilon}{1+\varepsilon}\beta_0(\Omega).$$

Interchanging the roles of the mappings and using (73) leads to

$$\beta_0(\Omega) \geq \frac{1-\varepsilon}{1+\varepsilon}\beta_0(\tilde{\Omega}).$$

The latter two inequalities together prove the theorem. □

Remark 2.57 Denotes η the conformal mapping of Ω onto $\tilde{\Omega}$ as above, then $\tilde{g}'(z) = \eta'(w) \cdot g'(z)$ (where $w = g(z)$) and the condition (70) is equivalent to

$$|\eta'(w) - 1| \le \varepsilon < 1, \text{ for all } w \in \Omega.$$

This means that the maximum norm of $\eta' - 1$ is small, which implies that η is near to the identity mapping. □

Remark 2.58 If $0 < \varepsilon < \frac{1}{4}$ then simple calculation shows

$$0 < 1 - 4\varepsilon \le \frac{1-\varepsilon}{1+\varepsilon} \text{ and } \frac{1+\varepsilon}{1-\varepsilon} \le 1 + 4\varepsilon.$$

Combining this with (71) gives

$$0 < (1 - 4\varepsilon)\beta_0(\Omega) \le \beta_0(\tilde{\Omega}) \le (1 + 4\varepsilon)\beta_0(\Omega),$$

which implies

$$|\beta_0(\tilde{\Omega}) - \beta_0(\Omega)| \le 2\sqrt{2}\varepsilon$$

using the obvious inequality $0 < \beta_0(\Omega) < \frac{1}{\sqrt{2}}$ (c.f. [46]). □

Remark 2.59 The numerical utility of Theorem 2.56 is the following. One constructs the univalent function $\tilde{g}$ such that it is for example a polynomial or a fractional rational function which approximates the function g in the described way. Then one can obtain $\beta_0(\tilde{\Omega})$ numerically solving a simpler, finite dimensional eigenvalue problem. With this value one has an estimate for the inf-sup constant of the domain. □

Remark 2.60 Let the domain Ω have $C^{1,\alpha}$ smooth boundary for some $0 < \alpha < 1$. Let the sequence of domains Ω_n, $n = 1, 2, \ldots$, tend to the domain Ω in the sense that for their corresponding conformal mappings one has the following: for every $0 < \varepsilon < 1$ there exists a natural number N such that

$$\sup_D |g_n' - g'| < \varepsilon, \tag{74}$$

whenever $n > N$. Then by Theorem 2.56 there follows

$$\lim_{n\to\infty} \beta_0(\Omega_n) = \beta_0(\Omega),$$

i.e. one has the convergence of the inf-sup constants. Therefore we have obtained a sufficient condition for the convergence of the inf-sup constants of a convergent domain sequence to the inf-sup constant of the limit domain. If one has a condition weaker than (74), then the convergence of the inf-sup constants can not be guaranteed, see Remark 2.65 below. □

Example 2.61 As a special case of Theorem 2.56 we can consider domains $\Omega = g(D)$ for which

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

such that $\sum_{n=2}^{\infty} n|a_n| \leq \varepsilon < 1$. Such domains are schlicht and nearly circular in the sense that

$$\pi \leq |\Omega| \leq \pi(1+\varepsilon) \text{ and } 2\pi \leq |\partial\Omega| \leq 2\pi(1+\varepsilon),$$

where $|\Omega|$ denotes the area of Ω and $|\partial\Omega|$ denotes the length of the closed curve $\partial\Omega$. We can choose $\tilde{g}(z) \equiv z$ and then there follows

$$\beta_0(g(D)) \geq \frac{1}{\sqrt{2}} \cdot \frac{1-\varepsilon}{1+\varepsilon}.$$

□

2.8 Polynomial mappings

In this section we investigate the case of polynomial mappings and give also some examples. As seen above in Section 2.4, for such domains the Schur complement operator is unitarily equivalent to the operator $\frac{1}{2}(\mathcal{I} - \mathcal{CMC})$, acting on the Hilbert space of sequences $\ell_{(2,-1)}$, where $\mathcal{M}$ is a finite matrix, see Corollary 2.21. For such domains it is possible to calculate the inf-sup constant: one must solve an eigenvalue problem for the matrix $\mathcal{M}$, choose the appropriate eigenvalue and use Remark 2.36.

Remark 2.62 A simple upper estimation for the inf-sup constant of the domain $g(D)$ is directly available if $g(z) = z + a_2 z^2 + \ldots a_n z^n$ is a univalent polynomial in D of order $\rho \geq 2$. For such polynomials we have $|a_n| \leq \frac{1}{n}$ (see [20]), and from the structure of the finite matrix $\mathcal{M}$ we have that

$$|\det \mathcal{M}| = n!|a_n|^n \leq \frac{n!}{n^n} < \frac{1}{2},$$

which leads to $|\mu_2|^{n-1} \geq n!|a_n|^n$ and by Remark 2.36 to the inequality

$$\beta_0^2(g(D)) \leq \frac{1}{2}\left(1 - (n!|a_n|^n)^{\frac{1}{n-1}}\right).$$

By $0 < |a_n| \leq \frac{1}{n}$ we have for the term constituting this upper estimate of the square of the inf-sup constant the inclusion

$$\frac{1}{4} \leq \frac{1}{2}\left(1 - \left(\frac{n!}{n^n}\right)^{\frac{1}{n-1}}\right) \leq \frac{1}{2}\left(1 - (n!|a_n|^n)^{\frac{1}{n-1}}\right) < \frac{1}{2}.$$

□

Next we give some examples.

Example 2.63 In case $g(z) = a_1 z + a_2 z^2$, $a_1 \neq 0$ we have $b_0 = \frac{1}{a_1}$ and $b_1 = -\frac{2a_2}{a_1^2}$ and

$$\mathcal{M} = \begin{pmatrix} \frac{|a_1|^2 - 2|a_2|^2}{\bar{a}_1^2} & \frac{a_2}{\bar{a}_1} \\ 2\frac{a_2}{\bar{a}_1} & 0 \end{pmatrix}.$$

The eigenvalues of $\mathcal{MC}$ are

$$\mu_1 = 1 \quad \text{with the eigenvector} \quad (a_1, 2a_2)^T,$$
$$\mu_2 = \frac{-2a_2^2}{\bar{a}_1^2} \quad \text{with the eigenvector} \quad (\bar{a}_2, -\bar{a}_1)^T.$$

Because $g'(z) \neq 0$ in $|z| \leq 1$ and using $g'\left(-\frac{a_1}{2a_2}\right) = 0$ there follows $\left|\frac{a_1}{2a_2}\right| > 1$. This gives $|\mu_2| < \frac{1}{2}$. Moreover for $a_1 = 1$ and $|a_2| \to \frac{1}{2}$ we have $|\mu_2| \to \frac{1}{2}$. If $a_1 = 1$ and $|a_2| = \frac{1}{2}$, then $g'(z) \neq 0$ in D, but $g'(z_0) = 0$ with $g''(z_0) \neq 0$ for a $z_0 \in \partial D$. (In this case there is an inner angle of 2π – an internal cusp – at the point $g(z_0) \in \partial\Omega$.) We have now $\beta_0^2(\Omega) = \frac{1}{2} - |\frac{a_2}{a_1}|^2$ for the inf-sup stability constant from Remark 2.36. □

Example 2.64 In case $g_{(m,\alpha)}(z) = z - \frac{c}{m^{\alpha+1}} z^m$ ($c \in \mathbb{R}$, $\alpha \geq 0$, $m > 1$ and integer), which is univalent in D for $|c| \leq m^\alpha$, we have

$$\mathcal{M} = \begin{pmatrix} 1 - \frac{c^2}{m^{2\alpha+1}} & 0 & \dots & 0 & -\frac{c}{m^{\alpha+1}} \\ 0 & 0 & \dots & -\frac{2c}{m^{\alpha+1}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\frac{(m-1)c}{m^{\alpha+1}} & \dots & 0 & 0 \\ -\frac{mc}{m^{\alpha+1}} & 0 & \dots & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\mathcal{MC}$ are now real. If $m \geq 2$ is even, then they are

$$1, \frac{c^2}{m^{2\alpha+1}} \text{ and } \pm\sqrt{k(m-k+1)}\frac{c}{m^{\alpha+1}} \text{ for } k = 2, \dots, \frac{m}{2}$$

For an odd $m \geq 3$, the eigenvalues are

$$1, \frac{c^2}{m^{2\alpha+1}}, \frac{m+1}{2} \cdot \frac{c}{m^{\alpha+1}},$$

and only for $m \geq 5$ odd we also have the eigenvalues

$$\pm\sqrt{k(m-k+1)}\frac{c}{m^{\alpha+1}} \text{ for } k = 2, \dots, \frac{m-1}{2}.$$

We have the inf-sup constant of $\Omega_{(m,\alpha)} := g_{(m,\alpha)}(D)$ from Remark 2.36:

$$\beta_0^2(\Omega_{(m,\alpha)}) = \begin{cases} \frac{1}{2}\left(1 - \frac{m+1}{2} \cdot \frac{|c|}{m^{\alpha+1}}\right) & \text{for } m \text{ odd,} \\ \frac{1}{2}\left(1 - \sqrt{\frac{m}{2}\left(\frac{m}{2}+1\right)}\frac{|c|}{m^{\alpha+1}}\right) & \text{for } m \text{ even.} \end{cases} \tag{75}$$

Moreover, the above $\beta_0^2(\Omega_{(m,\alpha)})$ are simple eigenvalues of the Schur complement operator for odd m and double eigenvalues of the Schur complement operator for even m. For example in the special case $m := 4$, $\alpha := 0$ and $c := 1$ one easily calculates that $\beta_0^2(\Omega_{(4,0)}) = \frac{1}{2} - \frac{\sqrt{6}}{8}$ is a double eigenvalue of $\mathcal{S}$ and the corresponding eigenfunctions are

$$p_1(z) = \operatorname{Re}\frac{1 + \frac{\sqrt{6}}{2}z^2}{1 - \frac{1}{4}z^3} \text{ and } p_2(z) = \operatorname{Im}\frac{1 - \frac{\sqrt{6}}{2}z^2}{1 - \frac{1}{4}z^3}.$$

□

Remark 2.65 We compute from the preceding example for $\alpha := 0$ and $0 < |c| \leq 1$

$$\begin{aligned} \|g_m - g\|_0^2 &= \frac{|c|^2}{m^2}\int_D z^m \bar{z}^m dxdy = \frac{\pi|c|^2}{m^2(m+1)}, \\ \max_{z\in D}|g_m(z) - g(z)| &= \frac{|c|}{m}\max_{z\in D}|z^m| = \frac{|c|}{m}, \\ \|g_m' - g'\|_0^2 &= \frac{\pi|c|^2}{m}, \\ \max_{z\in D}|g_m'(z) - g'(z)| &= |c|\max_{z\in D}|z|^{m-1} = |c|, \end{aligned}$$

where we have set $g_m := g_{(m,0)}$ and $g(z) = z$. These equalities show that $\lim_{m\to\infty} g_m = g$ in the L_2 and maximum norm on D, further $\lim_{m\to\infty} g_m' = g'$ is valid in the L_2 norm (but not in the maximum norm). In this sense we have a sequence of domains Ω_m, $m = 1, 2, \ldots$ converging to the unit disc D. The limit of the inf-sup constants (75) of the domains is however

$$\lim_{m\to\infty}\beta_0^2(\Omega_m) = \frac{1}{2} - \frac{|c|}{4} < \frac{1}{2} = \beta_0^2(D).$$

If we set $\alpha := 1$ into the Example 2.64, i.e. we investigate

$$\hat{g}_m(z) := g_{(m,1)}(z) = z - \frac{c}{m^2}z^m,$$

which is also univalent for $c \in \mathbb{R}$, $0 < |c| \leq 1$, then we get

$$\beta_0^2(\Omega) = \begin{cases} \frac{1}{2}\left(1 - \frac{m+1}{2m^2}|c|\right) & \text{for } m \text{ odd,} \\ \frac{1}{2}\left(1 - \frac{\sqrt{m(m+2)}}{2m^2}|c|\right) & \text{for } m \text{ even.} \end{cases}$$

We obtain in this modified case $\hat{g}_m \to g$ and $\hat{g}'_m \to g'$ for $m \to \infty$ in both (L_2 and maximum) norms. We also have $\lim_{m\to\infty} \beta_0^2(\hat{\Omega}_m) = \beta_0^2(D)$.

Our polynomial examples may seem far from practical relevance, but they cleairfy important questions: they show that the inf-sup constant of a domain can be also a multiple eigenvalue and show also the fact that neither the convergence of the mapping function nor the convergence of its derivative in the L_2 norm are sufficient to the convergence of the inf-sup constants of the domains to that of the limit domain. □

2.9 Estimations for other special classes of domains

In this section we examine univalent conformal, not necessarily polynomial, mappings g , which have special properties and estimate the inf-sup constants of the domains $\Omega = g(D)$. We assume that the univalent conformal mapping g of D onto Ω is normalized such that $g(0) = 0$ and $g'(0) = 1$. Let us denote the class of such functions by A. We define further subclasses of A. Let be $0 \le \alpha_1 < \alpha_2 \le \infty$ and define

$$S_{(\alpha_1,\alpha_2)} := \left\{ g \in A : \alpha_1 < \operatorname{Re} \frac{zg'(z)}{g(z)} \le \alpha_2 \right\} \tag{76}$$

Remark 2.66 If $\alpha_2 = \infty$ and $0 \le \alpha := \alpha_1 < 1$, then (76) alters to

$$S_\alpha := \left\{ g \in A : \operatorname{Re} \frac{zg'(z)}{g(z)} > \alpha \right\},$$

which is called the class of starlike functions of order α. If, moreover, $\alpha = 0$, then the functions belonging to S_0 map the unit disc D onto a starlike domain with respect to the origin, i.e. onto a domain for which from $w \in \Omega$ there follows $tw \in \Omega$ for all $0 \le t \le 1$. (Unfortunately, for other values of α there is no simple geometric representation.) □

A more convenient geometric description is available for domains which are maps of D under a function belonging to the class

$$S_\alpha^* := \left\{ g \in A : \left| \arg \frac{zg'(z)}{g(z)} \right| < \frac{\pi\alpha}{2} \right\}$$

for some $0 < \alpha \le 1$. S_α^* is the class of strongly starlike functions of order α. For $z \in \partial D$ the quantity $\arg(zg'(z))$ is the angle between the unit outward normal to $\partial g(D)$ at the point $g(z)$. Hence $\arg \frac{zg'(z)}{g(z)}$ equals the angle between the ray emitted from the origin to the point $g(z)$ and the unit outward normal to $\partial g(D)$ at the point $g(z)$. (The same angle appears in [46].) Thus a function from S^*_α, $\alpha < 1$ maps the unit disc onto such a bounded Jordan domain starlike with respect to the origin whose boundary is reachable from outside by the radial angle $\pi(1-\alpha)$. We have $S_0 = S^*_1$.

Another subclass of A is

$$K_{(\alpha_1,\alpha_2)} := \left\{g \in A : zg'(z) \in S_{(\alpha_1,\alpha_2)}\right\}. \tag{77}$$

Similarly as in Remark 2.66 we denote by $K_\alpha := \{g \in A : zg'(z) \in S_\alpha\}$ the class of convex functions of order α and by $K^*_\alpha := \{g \in A : zg'(z) \in S^*_\alpha\}$ the class of strongly convex functions of order α. Convex functions, i.e. those functions belonging to $K := K_0$, map D onto a convex domain. Moreover, it is known that $g \in K \Leftrightarrow zg' \in S$ and that every convex functions belongs to $S_{\frac{1}{2}}$, see e.g. [20].

First of all note that an estimation for the constant in the ineqality (54) is derived in [26] for star-shaped domains, which is used in [46] to obtain a lower estimation for the inf-sup constant.

Let $g \in S_0$ map the unit disc onto a domain Ω the boundary of which is represented in polar coordinates by $f(\varphi)e^{i\varphi}$, $0 \le \varphi < 2\pi$. Thus we have for $0 \le \theta < 2\pi$

$$g(e^{i\theta}) = f(\varphi(\theta))e^{i\varphi(\theta)},$$

where $\varphi = \varphi(\theta)$ is the boundary correspondence function. One easily obtains

$$\frac{zg'(z)}{g(z)} = \varphi'(\theta)\left(1 - i\frac{\dot f(\varphi)}{f(\varphi)}\right) \text{ for } z = e^{i\theta},$$

where $\dot f(\varphi) := \frac{df(\varphi)}{d\varphi}$ and $\varphi'(\theta) := \frac{d\varphi(\theta)}{d\theta}$ and therefore

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = \varphi'(\theta) \text{ and } \arg\frac{zg'(z)}{g(z)} = \arg\left(1 - i\frac{\dot f(\varphi)}{f(\varphi)}\right) = -\arctan\left(\frac{\dot f(\varphi)}{f(\varphi)}\right).$$

Now $g \in S_0$ implies $\varphi'(\theta) > 0$ and there also follows

$$|g'(z)| = |g(z)|\varphi'(\theta)\sqrt{1 + \left(\frac{\dot f(\varphi)}{f(\varphi)}\right)^2} \tag{78}$$

for $z \in \partial D$. We immediately have the following

Corollary 2.67 *Let the domain Ω be such that its corresponding conformal map is in the class (76) for some $0 < \alpha_1 < \alpha_2 < \infty$. Suppose further that* *the boundary of Ω lies in an annulus centered in the origin with inner radius r and outer radius R. Then we have for the inf-sup constant of the domain*

$$\beta_0(\Omega) \geq c_{\partial\Omega} \frac{r}{R},$$

where $c_{\partial\Omega} = \frac{\alpha_1}{\sqrt{2}\alpha_2} \left(\max_{0 \leq \varphi < 2\pi} \sqrt{1 + \left(\frac{\dot{f}(\varphi)}{f(\varphi)} \right)^2} \right)^{-1}$.

Proof. We use Corollary 2.46 along with equation (78) and definition (76). □

Remark 2.68 If we further suppose that $g \in S_{(\alpha_1, \alpha_2)} \cap S^*_\alpha$ for some $0 < \alpha_1 < \alpha_2 < \infty$ and $0 < \alpha < 1$, then there follows

$$1 \leq \sqrt{1 + \left(\frac{\dot{f}(\varphi)}{f(\varphi)} \right)^2} \leq \frac{1}{\cos \frac{\alpha\pi}{2}},$$

which gives the simpler expression $c_{\partial\Omega} = \frac{\alpha_1 \cos \frac{\alpha\pi}{2}}{\sqrt{2}\alpha_2}$ for the constant in the previous Corollary 2.67. □

Remark 2.69 The result given in Corollary 2.67 is not new, see e.g. [17], where also a similar result is mentioned about higher dimensional star-shaped domains. However, we provide in Corollary 2.67 also an explicit formula for the constant $c_{\partial\Omega}$ involved in the estimation depending on properties of the conformal mapping of the domain. □

Remark 2.70 The result given in Corollary 2.67 can also be related to another estimation given in [26], although only in case $\partial\Omega$ is smooth enough, see [54]. □

Remark 2.71 Certainly the domain dependence of the constant $c_{\partial\Omega}$ is hard to explain because of the boundedness condition imposed on the derivative of the boundary correspondence function φ. However, we mention that for example if Ω is convex then $\alpha_1 = \frac{1}{2}$ can be substituted. For a similar estimation see [54]. □

Example 2.72 The conditions of Corollary 2.67 are certainly fulfilled if

$$\left| \frac{zg'(z)}{g(z)} - \frac{1+\varepsilon^2}{1-\varepsilon^2} \right| \leq \frac{2\varepsilon}{1-\varepsilon^2}$$

for some $0 < \varepsilon < 1$. For such univalent conformal mappings we have

$$\frac{1-\varepsilon}{1+\varepsilon} \leq |\varphi'(\theta)| \leq \frac{1+\varepsilon}{1-\varepsilon} \text{ and } \left|\arg \frac{zg'(z)}{g(z)}\right| \leq \arcsin \frac{2\varepsilon}{1+\varepsilon^2}.$$

The lower estimation of the inf-sup constant from Corollary 2.46 becomes

$$\beta_0(\Omega) \geq \frac{1}{\sqrt{2}} \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2.$$

□

Remark 2.73 One can obtain an estimation of the inf-sup constant without the use of the boundary correspondence function. First observe that $g \in S^*_\alpha$ implies the subordination

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha,$$

that is, there exists a holomorphic function ω in D with $\omega(0) = 0$ and $|\omega(z)| \leq |z|$, $z \in D$, such that

$$\frac{zg'(z)}{g(z)} = \left(\frac{1+\omega(z)}{1-\omega(z)}\right)^\alpha$$

Consider a variant of inequality (54) using the positive (non-radial) weight function $\varrho(z) = \left|\frac{1+\omega(z)}{1-\omega(z)}\right|^{2\alpha}$, i.e. find a positive constant Γ_ϱ such that

$$\int_D u^2 \varrho dA \leq \Gamma_\varrho \int_D v^2 \varrho dA \tag{79}$$

for all conjugate harmonic functions u and v in $L_{2,\varrho}(D)$ satisfying (57) on D, where $L_{2,\varrho}(D) := \left\{u : \int_D u^2 \varrho dA < \infty\right\}$. The inequality (79) corresponds to a weighted variant of the Friedrichs inequality on D if we set $f = u + iv$:

$$\int_D f^2 \varrho dA \leq \gamma_\varrho \int_D |f|^2 \varrho dA,$$

where $\gamma_\varrho = \frac{\Gamma_\varrho - 1}{\Gamma_\varrho + 1}$ and the function f belongs to the subspace $AL_{2,\varrho}(D)$ of $L_{2,\varrho}(D)$ consisting of holomorphic functions. If such a constant $\Gamma_\varrho < \infty$ exists, then there follows similar as in Corollary 2.46

$$\Gamma_\Omega \leq \Gamma_\varrho \frac{R^2}{r^2},$$

where $0 < r \le |g(z)| \le R$ for $z \in \partial D$, that is, Ω is a bounded star-shaped domain whose boundary lies in an annulus with inner and outer radii r and R. For such a domain there follows

$$\beta_0(\Omega) \ge c_\varrho \frac{r}{R},$$

where the constant $c_\varrho := (1 + \Gamma_\varrho)^{-1/2}$ depends on the weight function, that is, on the shape of the domain and on the constant α, which depends also on the shape of the domain. □

An estimation for the inf-sup constant is also available for a domain Ω whose corresponding conformal mapping is in the class (77). If $\kappa(z)$ denotes the curvature of $\partial\Omega$ at the point $g(z)$, then we have

$$\kappa(z) = \frac{1}{|g'(z)|} \operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right).$$

Now, $g \in K_{(\alpha_1,\alpha_2)}$ implies

$$\frac{\alpha_1}{\kappa(z)} \le |g'(z)| \le \frac{\alpha_2}{\kappa(z)}.$$

By Corollary 2.46 there follows

Corollary 2.74 *If the domain Ω is such that its corresponding conformal map is in the class (77) for some $0 < \alpha_1 < \alpha_2 < \infty$, then we have for the inf-sup constant of the domain*

$$\beta_0(\Omega) \ge \frac{\alpha_1 \kappa_{\min}}{\sqrt{2}\alpha_2 \kappa_{\max}},$$

if $0 < \kappa_{\min}$ and $\kappa_{\max} < \infty$ are the minimal and maximal values of the curvature of $\partial\Omega$, respectively. □

2.10 Domains with corners

In this section we examine the case when the conformal map Ω of the unit disc has piecewise smooth boundary with a finite number of corners. For such domains the Friedrichs inequality holds, although the corresponding Friedrichs operator is not compact. This makes the estimation of the inf-sup constant difficult, see e.g. [46] for the case of a rectangle. We use the following known result (see [51]):

Proposition 2.75 *Let $\partial\Omega$ be a closed piecewise Lyapunov curve with the corners $g(b_k)$, $k = 1, \ldots, n$, where the interior angles are $\alpha_k\pi$, $0 < \alpha_k \leq 2$. If g is the conformal mapping of the unit disc D onto Ω such that $g'(0) > 0$, then*

$$g'(z) = h(z) \prod_{k=1}^{n} (z - b_k)^{\alpha_k - 1}, \tag{80}$$

where h is a Hölder continuous function on $\bar{D}$ different from zero. □

If we set $h \equiv 1$ on D, then (80) reduces to the well known Schwarz-Christoffel mapping of the unit disc onto a polygon with corners at $g(b_k)$ and with the correponding inner angles, see [37].

Let $\tilde{\Omega}$ denote a polygon with the corresponding Schwarz-Christoffel mapping $\tilde{g}$, which has the derivative

$$\tilde{g}'(z) = \prod_{k=1}^{n} (z - b_k)^{\alpha_k - 1}, \tag{81}$$

thus the polygon $\tilde{\Omega}$ has the same prevertices b_k, $k = 1, \ldots, n$ on ∂D as the domain Ω in Theorem 2.75. We have $g'(z) = \tilde{g}'(z)h(z)$, where by the Hölder continuity

$$1 \leq c_h := \frac{\sup_{\bar{D}} |h|}{\inf_{\bar{D}} |h|} < \infty.$$

We start again with the inequality (54) on $\tilde{\Omega}$ and calculate similar as in Theorem 2.51. There follows

$$\Gamma_\Omega \leq c_h^2 \Gamma_{\tilde{\Omega}},$$

which, on the other hand, implies by (59)

$$\beta_0(\Omega) \geq \frac{1}{c_h} \beta_0(\tilde{\Omega}). \tag{82}$$

Corollary 2.76 *Let g and $\tilde{g}$ be the conformal mappings of the unit disc onto the domains Ω and $\tilde{\Omega}$ such that their derivatives are* (80) *and* (81)*. If the domain Ω has a piecewise Lyapunov boundary, then there follows* (82) *with some constant $1 \leq c_h < \infty$.*

In the following we calculate some examples.

Example 2.77 We investigate the conformal map

$$g(z) = (1 - z)^{\omega}, \tag{83}$$

$0 < \omega \leq 2$ of the unit disc onto a ("drop shaped") finite domain $\Omega^{(\omega)}$ with smooth boundary except at $g(1) = 0$, where it has an inner angle $\omega\pi$ for $\omega \neq 1$. If $\omega = 1$ then the domain is the translated unit disc. (In case $\omega = 2$ the domain is a cardioid and it has an interior cusp at the origin.) Using the series expansions

$$g(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k-\omega)}{\Gamma(-\omega)\Gamma(k+1)} z^k \text{ and } \frac{1}{g'(z)} = \sum_{k=0}^{\infty} \frac{-\Gamma(k-1+\omega)}{\omega\Gamma(\omega-1)\Gamma(k+1)} z^k$$

for (83) we calculate for $k = 0, 1, 2, \ldots$

$$s_k^{(\omega)} = \frac{\sin(\omega\pi)}{\omega\pi} \left(\frac{1}{(k-\omega+1)} - \frac{1}{(k-\omega+2)} \right).$$

Set $\mathcal{F}_D^{(\omega)}$ for the transformed Friedrichs operator of the domain with the associated mapping (83). A simple calculation with the associated infinite matrices shows

$$\left(\mathcal{F}_D^{(1+\omega)}(M_z p), M_z p \right) = \frac{\omega}{1+\omega} \left(\mathcal{F}_D^{(\omega)} p, M_z p \right).$$

for $p \in L_2(D)$ and $0 \leq \omega \leq 1$, where M_z denotes the multiplication by the variable z. For $p(z) = \sum_{k=0}^{\infty} p_k z^k \in AL_2(D)$ we have

$$\|p\|^2 = \sum_{k=0}^{\infty} \frac{1}{k+1} |p_k|^2 \text{ and } \|M_z p\|^2 = \sum_{k=0}^{\infty} \frac{1}{k+2} |p_k|^2.$$

There follows $\|M_z p\|^2 \leq \|p\|^2 \leq 2\|M_z p\|^2$. Using this along with the Cauchy-Schwarz inequality, gives:

$$\frac{\left| \left(\mathcal{F}_D^{(1+\omega)}(M_z p), M_z p \right) \right|}{\|M_z p\|^2} \leq \left| \frac{\sqrt{2}\omega}{1+\omega} \right| \cdot \frac{\|\mathcal{F}_D^{(\omega)} p\|}{\|p\|}$$

According to $\|\mathcal{F}^{(\omega)}\| \leq 1$ in operator norm we obtain

$$\|\mathcal{F}^{(1+\omega)}\| \leq \left| \frac{\sqrt{2}\omega}{1+\omega} \right| < 1 \tag{84}$$

for $0 \leq \omega \leq 1$, which implies

$$\beta_0^2(\Omega^{(1+\omega)}) \geq \frac{1}{2} \left(1 - \left| \frac{\sqrt{2}\omega}{1+\omega} \right| \right). \tag{85}$$

□

Remark 2.78 The inequality (84) holds also for $1-\sqrt{2}\le\omega<0$ but then the domain $\Omega^{(\omega)}$ is infinite although its Friedrichs operator also satisfies the norm estimate $\|\mathcal{F}^{(\omega)}\|\le 1$. Therefore the estimation (85) is also valid for all domains $\Omega^{(1+\omega)}$ with $1-\sqrt{2}\le\omega\le 0$. □

Example 2.79 Another example for a domain with a corner is a wedge:

$$W_\alpha := \{z\in\mathbb{C} : |\arg z| \le \frac{\alpha}{2}\}$$

This domain has a corner at the origin and hence the derivative of the associated conformal map has a singularity on the unit circle. A simple calculation of the quantities (35) shows

$$s_{2k}^{(W_\alpha)} = \frac{\sin\alpha}{\alpha}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right); \qquad s_{2k+1}^{(W_\alpha)} = 0,$$

for $k=0,1,\ldots$, and further

$$\mathcal{M}^{(W_\alpha)} = \frac{\sin\alpha}{\alpha}\mathcal{M}^{(W_0)},$$

where the set $W_0 := \left\{z\in\mathbb{C} : |\operatorname{Im} z| \le \frac{\pi}{2}\right\}$ is an infinite strip. Therefore we have a similar equality for all eigenvalues and there follows

$$1-2\beta_0^2(W_\alpha) = \frac{\sin\alpha}{\alpha}\left(1-2\beta_0^2(W_0)\right)$$

By $0<\beta_0^2(W_0)\le\frac{1}{2}$ (see e.g. [39]) we obtain

$$\frac{1}{2}\left(1-\left|\frac{\sin\alpha}{\alpha}\right|\right) \le \beta_0^2(W_\alpha) \le \frac{1}{2}.$$

Moreover in [39] it is proved that the infinite strip and the wedge are quadrature domains in the generalized sense and the concerned operators have only continuous spectra. □

Example 2.80 Consider the conformal map

$$g(z) = \frac{z}{\sqrt{1+z^2}},$$

which maps the unit disc D onto the exterior of a hyperbola (i.e. that region which does not contain the foci of the hyperbola). We have

$$g(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(k+1\right)}z^{2k+1} \text{ and } \frac{1}{g'(z)} = \sum_{k=0}^{\infty}\frac{3(-1)^k\Gamma\left(k-\frac{3}{2}\right)}{4\sqrt{\pi}\Gamma\left(k+1\right)}z^{2k},$$

which imply

$$s_{2k} = (-1)^k \frac{2}{\pi}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) \text{ and } s_{2k+1} = 0,$$

for $k = 0, 1, \ldots$. These are almost the same quantities as in the case of a wedge having a corner with $\alpha = \frac{\pi}{2}$ (see the previous Example 2.79). An examination of the corresponding matrices gives:

$$\mathcal{M}^{(W_\alpha)} p = \mu p \quad \Leftrightarrow \quad \mathcal{M}^{(\text{hyperbola})} \hat{p} = \mu \hat{p},$$
$$\mathcal{M}^{(W_\alpha)} q = \mu q \quad \Leftrightarrow \quad \mathcal{M}^{(\text{hyperbola})} \hat{q} = -\mu \hat{q},$$

for those eigenvectors having the entries

$$\hat{p}_{2k} = (-1)^k p_{2k}, \hat{p}_{2k+1} = p_{2k+1} = 0 \text{ and } \hat{q}_{2k} = q_{2k} = 0, \hat{q}_{2k+1} = (-1)^k q_{2k+1},$$

for $k = 0, 1, \ldots$. Therefore we have obtained

$$\beta_0(W_{\frac{\pi}{2}}) = \beta_0(\text{hyperbola}).$$

□

2.11 Case of multiply connected domains

All the previous results are valid certainly only for simply connected domains. The next result from [22] suggests some correspondence between the operators $\mathcal{F}$ and $\mathcal{S}$, at least in cases of smoothly enough bounded domains. This result connects the Friedrichs eigenvalue problem (17) to another problem of a quite different character.

Proposition 2.81 ***(Garabedian, [22])*** *Let the boundary of the domain Ω have continuous curvature. The function $f \in AL_{2,0}(\Omega)$ and $\mu \in \mathbb{R}$ satisfy the eigenvalue problem* (17) *if and only if the function defined by*

$$U(w) := \frac{1}{\pi} \int_\Omega \frac{\mu \overline{f(\omega)} - f(\omega)}{w - \omega} dA(\omega)$$

satisfies the eigenvalue problem

$$\partial_{\bar{w}}^2 U + \mu \partial_{\bar{w}} \partial_w \bar{U} = 0 \text{ in } \Omega \text{ and } U = 0 \text{ on } \partial\Omega. \tag{86}$$

Moreover we also have for $w \in \Omega$

$$\partial_{\bar{w}} U(w) = -\mu \overline{f(w)} + f(w) \tag{87}$$

which gives

$$(1 - \mu^2) f(w) = \partial_{\bar{w}} U(w) + \mu \partial_w \overline{U(w)}. \quad \square \tag{88}$$

□

Now let us clarify how (86) is connected to the eigenvalue problem of the Schur complement operator. As in the proof of Theorem 2.5 we use

$$\Delta = 4\partial_{\bar{w}}\partial_w, \operatorname{div} = 2\operatorname{Re}\partial_w \text{ and } \operatorname{grad} = 2\partial_{\bar{w}}.$$

For some complex valued function v there follows

$$\begin{aligned}\operatorname{grad}\operatorname{div} v &= 2\partial_{\bar{w}}(2\operatorname{Re}\partial_w v) = 2\partial_{\bar{w}}(\partial_w v + \overline{\partial_w v}) \\ &= 2\partial_{\bar{w}}\partial_w v + 2\partial_{\bar{w}}^2\bar{v} = \frac{1}{2}\Delta v + 2\partial_{\bar{w}}^2\bar{v},\end{aligned}$$

which gives

$$\partial_{\bar{w}}^2\bar{v} = \frac{1}{2}\operatorname{grad}\operatorname{div} v - \frac{1}{4}\Delta v$$

and

$$\partial_{\bar{w}}^2(2\bar{v}) + \mu\partial_{\bar{w}}\partial_w(2v) = \operatorname{grad}\operatorname{div} v - \frac{1-\mu}{2}\Delta v.$$

Thus U and μ solve (86) if and only if $v := \frac{1}{2}\bar{U}$ and $\lambda := \frac{1-\mu}{2}$ solve

$$\operatorname{grad}\operatorname{div} v - \lambda\Delta v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial\Omega,$$

which is called the Cosserat eigenvalue problem, see e.g. [49]. This is on the other hand equivalent to

$$\Delta v = \operatorname{grad} p, \operatorname{div} v = \lambda p \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial\Omega$$

with some real valued function p, which is the eigenvalue problem for the Schur complement operator of the first kind Stokes problem with the pressure eigenfunction p to the eigenvalue λ. Proposition 2.81 relates by equation (87) this pressure eigenfunction p to the Friedrichs eigenfunction f:

$$\begin{aligned}\lambda p &= \operatorname{div} v = 2\operatorname{Re}\partial_w v = 2\operatorname{Re}\overline{\partial_w v} = 2\operatorname{Re}\partial_{\bar{w}}\bar{v} \\ &= \operatorname{Re}\partial_{\bar{w}}U = \operatorname{Re}(f - \mu\bar{f}) = (1-\mu)\operatorname{Re} f = 2\lambda\operatorname{Re} f.\end{aligned}$$

From $f \in AL_{2,0}(\Omega)$ follows $0 < \mu < 1$ by Friedrichs inequality, which implies $\lambda \neq 0$. Thus we obtained $p = 2\operatorname{Re} f$. The same computation with $-if$ instead of f implies the pressure eigenfunction $2\operatorname{Im} f$ to the eigenvalue $\frac{1+\mu}{2}$. We have, similar to the case of simply connected domains, that a Friedrichs eigenvalue $0 < \mu < 1$ to the eigenfunction f implies two eigenvalues $\frac{1\mp\mu}{2}$ (symmetric with respect to $\frac{1}{2}$) with the corresponding pressure eigenfunctions $2\operatorname{Re} f$ and $2\operatorname{Im} f$.

Next we want to show a correspondence between the operators $\mathcal{F}$ and $\mathcal{S}$ for multiply connected domains as suggested by the above result. The usage of conformal mapping like in section 2.3 is impossible because in the case of multiply connected domains there is no canonical domain like the unit disc for simply connected domains, see [37]. Proposition 2.4 is proved in [40] only for simply connected domains, hence we do not have this possibility (as in section 2.2) either. The main tool for proving a correspondence like Theorem 2.5 in case of multiply connected domains is the representation result Proposition 3.2 below in complex formulation (see Remark 3.3) and the result given in [22]:

$$K(w,\omega) = -\frac{2}{\pi}\partial_w \partial_{\bar{\omega}} G(w,\omega), \tag{89}$$

where $K(w,\omega)$ is the Bergman kernel and $G(w,\omega)$ denotes the Green's function of the domain Ω of finite connectivity, the boundary curves of which have continuous curvature. The technique of the proof is very similar to that of 2.5. Let h be a harmonic function in Ω. We define $F(w) := \partial_w h(w)$ which is holomorphic because h is harmonic:

$$\partial_{\bar{w}} F(w) = \partial_{\bar{w}} \partial_w h(w) = \frac{1}{4}\Delta h(w) = 0.$$

The functions

$$\begin{aligned} u_0(w) &:= -w\overline{F(w)} = -w\overline{\partial_w h(w)}, \\ p_R(w) &:= -4\,\mathrm{Re}\,F(w) = -4\,\mathrm{Re}\,\partial_w h(w) \end{aligned}$$

satisfy by Remark 3.3 the homogeneous momentum equation (5) and $\operatorname{div} u_0 = \frac{1}{2}p_R$. The harmonic function

$$H(w) := -\frac{1}{\pi i}\int_{\partial\Omega} u_0(\omega)\partial_{\bar{\omega}} G(w,\omega)d\bar{\omega}$$

has boundary values equal to the boundary values of $-u_0$, see [22]. Substituting the definition of u_0 and partial differentiation with respect to w gives

$$\partial_w H(w) = \frac{1}{\pi i}\int_{\partial\Omega} \partial_w \partial_{\bar{\omega}} G(w,\omega)\omega\overline{\partial_\omega h(\omega)}d\bar{\omega} = -\frac{1}{2i}\int_{\partial\Omega} K(w,\omega)\omega\partial_{\bar{\omega}}\overline{h(\omega)}d\bar{\omega},$$

where (89) has also been used. Green's theorem yields

$$\partial_w H(w) = \int_\Omega K(w,\omega)\omega\partial_\omega\partial_{\bar{\omega}}\overline{h(\omega)}dA(\omega) + \int_\Omega \partial_\omega(K(w,\omega)\omega)\overline{\partial_\omega h(\omega)}dA(\omega).$$

The first term equals zero because h is harmonic. We obtain therefore

$$\partial_w H(w) = \int_\Omega K(w,\omega)\overline{\partial_\omega h(\omega)}dA(\omega) = \int_\Omega K(w,\omega)\overline{F(\omega)}dA(\omega)$$

using also $\partial_\omega K(w,\omega) = 0$ for the Bergman kernel. Now the function

$$u = u_0 + H$$

satisfies the homogeneous momentum equation (5) with the corresponding pressure p_R and has zero boundary values. Thus we have

$$\mathcal{S}p_R(w) = \operatorname{div} u(w) = \frac{1}{2}p_R(w) + 2\operatorname{Re}\int_\Omega K(w,\omega)\overline{F(\omega)}dA(\omega).$$

A very similar calculation with $-ih$ instead of h gives

$$\mathcal{S}p_I(w) = \operatorname{div} u(w) = \frac{1}{2}p_I(w) - 2\operatorname{Im}\int_\Omega K(w,\omega)\overline{F(\omega)}dA(\omega),$$

where $-4F(w) = p_R(w) + ip_I(w)$. We compose the latter equations.

$$2\mathcal{S}F(w) = F(w) - \overline{\int_\Omega K(w,\omega)\overline{F(\omega)}dA(\omega)}.$$

Thus, similar to Theorem 2.5, we obtained the following

Theorem 2.82 *Let* Ω *be a domain of finite connectivity with boundary components having continuous curvature. Then* (19) *holds for the Schur complement and the Friedrichs operators of* Ω. □

3 Representation results

In this section we investigate the representation formulae given in [29] and [30] for Stokes flows via harmonic functions. We mainly avoid complex notation because three-dimensional domains are also concerned, but the connection with the preceding results is also shown. We prove in subsection 3.1.1 the equivalence of the different representations given in [29] and [30] for three-dimensional star-shaped domains and, moreover, we establish connection formulae between the used harmonic functions. In Section 3.1.2 we give a possible generalization for arbitrary three-dimensional domains. The connection to another representation from [38] is also explained. In Section 3.2 we generalize the representations for a wider class of functions, i.e. for the solutions of Navier's equation for linear elasticity. Finally the connection between the two and three-dimensional formulae is also explained.

This section is mainly based on the paper [55] of the author.

The notation is mostly similar to that used in [29] and [30]. We use the usual inner product and vector product in $\mathbb{R}^3$. In the two-dimensional formulae we set $\vec{x}^\perp = \binom{x_1}{x_2}^\perp := \binom{x_2}{-x_1}$. The differential operators ∇, div, rot, and Δ denote the gradient, divergence, rotation, and Laplacian of scalar functions respectively vector fields, where for a two-dimensional vector field $\vec{v} = (v_1, v_2)^T$ the rotation means the scalar $\operatorname{rot}\vec{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ as in the previous part of the thesis.

Definition 3.1 *A vector field $\vec{v}$ is called a Stokes function in a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, if $\vec{v} \in C_2(\Omega)$ and if there exists a (corresponding pressure-) function $p \in C_1(\Omega)$ such that*

$$\Delta\vec{v}(\vec{x}) = \nabla p(\vec{x}), \quad \operatorname{div}\vec{v}(\vec{x}) = 0 \quad \text{for} \quad \vec{x} \in \Omega. \tag{90}$$

Proposition 3.2 ***(Theorem 1 in [29])*** *Let $\Omega \subset \mathbb{R}^2$ be a domain. A function $\vec{v} : \Omega \to \mathbb{R}^2$ is a Stokes function with corresponding pressure p in Ω if and only if there exists a harmonic function $\vec{h}$ in Ω such that*

$$\vec{v}\,(\vec{x}) = -\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}\,(\vec{x}) + \vec{x}^\perp \operatorname{rot}\vec{h}\,(\vec{x})\right) + \vec{h}\,(\vec{x})\,, \tag{91}$$

$$p\,(\vec{x}) = -2\operatorname{div}\vec{h}\,(\vec{x}) \ \ \text{for } \vec{x} \in \Omega; \tag{92}$$

this harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}\,(\vec{x}) = \vec{v}\,(\vec{x}) - \frac{1}{4}\left(p\,(\vec{x})\,\vec{x} - \vec{x}^\perp \operatorname{rot}\vec{v}\,(\vec{x})\right) \ \ \text{for } \vec{x} \in \Omega. \tag{93}$$

□

Remark 3.3 Note, that the preceding Proposition 3.2 is valid for arbitrary plane domains and not only for simply connected ones. Moreover, in the case of simply connected domains it is equivalent to the representation in complex notation used in Proposition 2.10, because one easily verifies

$$-\frac{1}{2}\left(\vec{x}\,\mathrm{div}\,\vec{h}(\vec{x}) + \vec{x}^{\perp}\,\mathrm{rot}\,\vec{h}(\vec{x})\right) + \vec{h}(\vec{x}) = -z\overline{\partial_z h(z)} + h(z).$$

Note additionally that in simply connected domains each harmonic function h decomposes as $h = v_1 + \overline{v_2}$ with holomorphic functions v_1 and v_2, i.e.

$$-z\overline{\partial_z h(z)} + h(z) = -z\overline{v_1'(z)} + v_1 + \overline{v_2},$$

which shows the correspondence between the various representations of two-dimensional Stokes functions given in Proposition 3.2 and in Proposition 2.10 in the previous section 2. Hence, the investigation and generalization of these representation formulae connects this section naturally to the previous section. □

3.1 Stokes functions in the three-dimensional case

As a generalization of the representation for two-dimensional Stokes functions in Proposition 3.2 we have for three-dimensional Stokes functions

Proposition 3.4 ***(Theorem 2 in [29])*** *Assume that $\Omega \subset \mathbb{R}^3$ is a star-shaped domain with respect to the origin. Then a function $\vec{v} : \Omega \to \mathbb{R}^3$ is a Stokes function with corresponding pressure p in Ω if and only if there exists a harmonic function $\vec{h}$ in Ω such that*

$$\vec{v}(\vec{x}) = \frac{3}{2}\vec{h}(\vec{x}) - \frac{1}{2}\left(\vec{x}\,\mathrm{div}\,\vec{h}(\vec{x}) + \vec{x} \times \mathrm{rot}\,\vec{h}(\vec{x})\right), \tag{94}$$

$$p(\vec{x}) = -2\,\mathrm{div}\,\vec{h}(\vec{x}) \text{ for } \vec{x} \in \Omega; \tag{95}$$

this harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}(\vec{x}) = \frac{2}{3}\vec{v}(\vec{x}) - \frac{1}{6}\left(p(\vec{x})\vec{x} - \vec{x} \times \mathrm{rot}\,\vec{v}(\vec{x})\right) + \frac{2}{3}\vec{x} \times \nabla\phi(\vec{x}) \text{ for } \vec{x} \in \Omega, \tag{96}$$

where the function ϕ is harmonic in Ω and given by

$$\phi(\vec{x}) = -\frac{1}{4}\int_0^1 t^4 \vec{x} \cdot \mathrm{rot}\,\vec{v}(t\vec{x})dt \text{ for } \vec{x} \in \Omega. \tag{97}$$

□

Note, that in Proposition 3.4 we slightly changed the harmonic function $\tilde{\vec{v}}$ as used in [29]–Theorem 2; we use instead $\vec{h} = \frac{3}{2}\tilde{\vec{v}}$. Note further, that Proposition 3.4 is valid only for three-dimensional star-shaped domains.

A seemingly other representation theorem is given in

Proposition 3.5 *(Kratz-Lindae, [30].) Assume that $\Omega \subset \mathbb{R}^3$ is a star-shaped domain with respect to the origin. Then a function $\vec{v} : \Omega \to \mathbb{R}^3$ is a Stokes function with corresponding pressure p in Ω if and only if there exist (scalar) harmonic functions Ψ, Φ and φ in Ω such that*

$$\vec{v}(\vec{x}) = \nabla\varphi(\vec{x}) + rot\left(r^2\vec{x} \times \nabla\Psi(\vec{x})\right) + \vec{x} \times \nabla\Phi(\vec{x}), \text{ where } r = |\vec{x}|, \quad (98)$$

$$p(\vec{x}) = -6\Psi(\vec{x}) - 10\vec{x} \cdot \nabla\Psi(\vec{x}) - 4\vec{x} \cdot \nabla\left(\vec{x} \cdot \nabla\Psi(\vec{x})\right) \text{ for } \vec{x} \in \Omega. \quad (99)$$

These harmonic functions are uniquely determined by the Stokes function $\vec{v}$ (and its pressure p) under the normalization $p(0) = \Psi(0) = \Phi(0) = \varphi(0) = 0$, and they are given by the formulae $\Psi(\vec{x}) = \frac{1}{2}\int_0^1 \{\sqrt{\tau} - 1\} p(\tau\vec{x}) d\tau$, $\Phi(\vec{x}) = -\int_0^1 w(\tau\vec{x}) d\tau$, and $\varphi(\vec{x}) = \int_0^1 \vec{x} \cdot \tilde{\vec{v}}(\tau\vec{x}) d\tau$ for $\vec{x} \in \Omega$, where $\tilde{\vec{v}}(\vec{x}) = \vec{v}(\vec{x}) - rot\left(\vec{x} \times \nabla\left(r^2\Psi\right)\right)$, and $w(\vec{x}) = \vec{x} \cdot \int_0^1 rot\, \vec{v}^*(\tau\vec{x}) d\tau$, with $\vec{v}^*(\vec{x}) = \tilde{\vec{v}}(\vec{x}) - \nabla\varphi(\vec{x})$. □

3.1.1 Equivalent representations in star-shaped domains

In this subsection we show a way how the representation results in Propositions 3.4 and 3.5 can be unified. To achieve this we formulate the

Definition 3.6 *The scalar function $p \in C_1(\Omega)$ and the vector field $\vec{q} \in C_1(\Omega)$ are called conjugate if they satisfy the equations*

$$rot\, \vec{q}(\vec{x}) = -\nabla p(\vec{x}), \quad div\, \vec{q}(\vec{x}) = 0 \quad \text{for} \quad \vec{x} \in \Omega. \quad (100)$$

In quaternionic analysis, equations (100) are known as equations of Moisil-Teodorescu. They are also considered in [50] in connection with the Crouzeix-Velte-decomposition of vector functions (belonging to a Sobolev space) into divergence free and rotation free and biharmonic parts.

For the proofs of the results below we need several formulae from vector analysis, which were partly used in [29] and [30], too. Here we assume the existence and/or continuity of the partial derivatives of the functions $\varphi :=$

$\varphi(\vec{x})$ and $\vec{u} := \vec{u}(\vec{x})$ involved.

$$\begin{aligned}
\vec{x}\,\mathrm{div}\,\vec{u} + \vec{x}\times\mathrm{rot}\,\vec{u} &= \vec{u} + \nabla\left(\vec{x}\cdot\vec{u}\right) + \mathrm{rot}\left(\vec{x}\times\vec{u}\right) && (101)\\
\mathrm{div}\left(\vec{x}\times\vec{u}\right) &= -\vec{x}\cdot\mathrm{rot}\,\vec{u} && (102)\\
\Delta\left(\vec{x}\cdot\vec{u}\right) &= \vec{x}\cdot\Delta\vec{u} + 2\,\mathrm{div}\,\vec{u} && (103)\\
\Delta\left(\vec{x}\times\vec{u}\right) &= \vec{x}\times\Delta\vec{u} + 2\,\mathrm{rot}\,\vec{u} && (104)\\
\mathrm{rot}\left(\varphi\vec{x}\right) &= -\vec{x}\times\nabla\varphi && (105)\\
\mathrm{div}\left(\varphi\vec{x}\right) &= n\varphi + \vec{x}\cdot\nabla\varphi, \text{ for } \vec{x}\in\mathbb{R}^n && (106)\\
\Delta\left(\varphi\vec{x}\right) &= \vec{x}\Delta\varphi + 2\nabla\varphi && (107)\\
\Delta\left(\vec{x}\times\nabla\varphi\right) &= \vec{x}\times\nabla\Delta\varphi && (108)\\
\mathrm{rot}\left(\vec{x}\times\nabla\varphi\right) &= -\nabla\left(\varphi + \vec{x}\times\nabla\varphi\right) + \vec{x}\Delta\varphi && (109)\\
\nabla\left(r^2\varphi\right) &= 2\varphi\vec{x} + r^2\nabla\varphi, \text{ with } r = |\vec{x}| && (110)\\
\Delta\left(r^2\varphi\right) &= 6\varphi + 4\vec{x}\cdot\nabla\varphi + r^2\Delta\varphi && (111)\\
\vec{x}\Delta\left(r^2\varphi\right) &= \nabla\left(r^2\varphi + \vec{x}\cdot\nabla\left(r^2\varphi\right)\right) + \mathrm{rot}\left(r^2\vec{x}\times\nabla\varphi\right) && (112)
\end{aligned}$$

To prove the main result of this subsection we formulate a lemma which gives a representation of conjugate pairs in star-shaped domains similar to that in Proposition 3.5 for Stokes functions and which is of interest on its own.

Lemma 3.7 *Assume $\Omega\subset\mathbb{R}^3$ is a star-shaped domain with respect to the origin. Then the pair $(\vec{q},p)$ on Ω is a conjugate pair iff there exist scalar harmonic functions φ and ϕ in Ω such that*

$$\begin{aligned}
\vec{q}(\vec{x}) &= \nabla\varphi(\vec{x}) + \vec{x}\times\nabla\phi(\vec{x}), && (113)\\
p(\vec{x}) &= \phi(\vec{x}) + \vec{x}\cdot\nabla\phi(\vec{x}), && (114)
\end{aligned}$$

for $\vec{x}\in\Omega$. These harmonic functions are uniquely determined by the pair $(\vec{q},p)$ under the normalization $\varphi(0)=\phi(0)=p(0)=0$, and they are given by the formulae

$$\begin{aligned}
\phi(\vec{x}) &= \int_0^1 \frac{1}{\tau^2}e^{1-\frac{1}{\tau}}p(\tau\vec{x})d\tau, && (115)\\
\varphi(\vec{x}) &= \int_0^1 \vec{x}\cdot\vec{q}(\tau\vec{x})d\tau\left(=\int_0^1 \vec{x}\cdot\left(\vec{q}(\tau\vec{x}) - \tau\vec{x}\times\nabla\phi(\tau\vec{x})\right)d\tau\right). && (116)
\end{aligned}$$

Proof. From (105) and (109) we see that (113) and (114) satisfy (100) with harmonic φ and ϕ.

To prove the other direction, the following identities are useful (see (8) and (9) in [30]):

$$\tau \frac{d}{d\tau}\left\{\phi\left(\tau\vec{x}\right)\right\} = \vec{x}\cdot\nabla\left\{\phi\left(\tau\vec{x}\right)\right\}, \tag{117}$$

$$\tau \frac{d}{d\tau}\left\{\vec{q}\left(\tau\vec{x}\right)\right\} = -\vec{q}\left(\tau\vec{x}\right)+\nabla\left\{\vec{x}\cdot\vec{q}\left(\tau\vec{x}\right)\right\}. \tag{118}$$

For fixed $\vec{x}\in\Omega$ set $z\left(t\right):=\phi\left(t\vec{x}\right)$. Using (114) and the identity (117) there follows $p\left(t\vec{x}\right)=\phi\left(t\vec{x}\right)+t\vec{x}\cdot\nabla\left\{\phi\left(t\vec{x}\right)\right\}$ and the ordinary differential equation for the function $z(t)$:

$$t^2 z'(t)+z(t)=p\left(t\vec{x}\right).$$

Using the initial condition $z(0)=0$ we solve it and get

$$z(t)=e^{\frac{1}{t}}\int_0^t \frac{1}{\tau^2}e^{-\frac{1}{\tau}}p\left(\tau\vec{x}\right)d\tau.$$

So we have $\phi(\vec{x})=z(1)=\int_0^1\frac{1}{\tau^2}e^{1-\frac{1}{\tau}}p(\tau\vec{x})d\tau$. Choosing this expression as the definition (115) for ϕ, by backward calculation we have (114). Moreover

$$\Delta\phi\left(\vec{x}\right)=\Delta\int_0^1\frac{1}{\tau^2}e^{1-\frac{1}{\tau}}p\left(\tau\vec{x}\right)d\tau=\int_0^1\frac{1}{\tau}e^{1-\frac{1}{\tau}}\Delta p\left(\tau\vec{x}\right)d\tau=0$$

because of $\Delta p=0$. Substituting $\tilde{\vec{q}}(\vec{x}):=\vec{q}(\vec{x})-\vec{x}\times\nabla\phi(\vec{x})$ into (118) we observe $\vec{x}\cdot\tilde{\vec{q}}(\vec{x})=\vec{x}\cdot\vec{q}(\vec{x})$, and therefore

$$\int_0^1\tau\frac{d}{d\tau}\left\{\tilde{\vec{q}}(\tau\vec{x})\right\}d\tau=-\int_0^1\tilde{\vec{q}}(\tau\vec{x})\,d\tau+\int_0^1\nabla\left\{\vec{x}\cdot\vec{q}\left(\tau\vec{x}\right)\right\}d\tau.$$

By interchanging integration and gradient we find (116) on the right-hand side. Now we integrate by parts:

$$\left[\tau\tilde{\vec{q}}(\tau\vec{x})\right]_0^1-\int_0^1\tilde{\vec{q}}(\tau\vec{x})\,d\tau=-\int_0^1\tilde{\vec{q}}(\tau\vec{x})\,d\tau+\nabla\varphi\left(\vec{x}\right),$$

we have $\tilde{\vec{q}}\left(\vec{x}\right)=\nabla\phi\left(\vec{x}\right)$ and (113) follows. Moreover by (103) and (100) we have

$$\Delta\varphi\left(\vec{x}\right)=\Delta\int_0^1\vec{x}\cdot\vec{q}\left(\tau\vec{x}\right)d\tau\int_0^1\tau^2\vec{x}\cdot\left(\Delta\vec{q}\right)\left(\tau\vec{x}\right)+2\tau\left(\operatorname{div}\vec{q}\right)\left(\tau\vec{x}\right)d\tau=0.$$

It remains to show the uniqueness of φ and ϕ. For this let φ and ϕ be harmonic with $\varphi(0)=\phi(0)=0$, such that $\vec{q}=\nabla\varphi+\vec{x}\times\nabla\phi\equiv 0$ and

$p = \phi + \vec{x} \cdot \nabla\phi \equiv 0$. Again by fixing $\vec{x} \in \Omega$ and setting $z(t) = \phi(t\vec{x})$ we have $z \in C^\infty[0,1]$ and from (117) there follows

$$t^2 z'(t) + z(t) = 0 \text{ and } z(0) = 0.$$

This ordinary differential equation has the unique solution $z(t) \equiv 0$ which implies $\phi \equiv 0$. Substituting this into the equation $\nabla\varphi + \vec{x} \times \nabla\phi \equiv 0$ we have $\nabla\varphi \equiv 0$ and by the normalization also $\varphi \equiv 0$. □

Theorem 3.8 *Let $\Omega \subset \mathbb{R}^3$ be a star-shaped domain with respect to the origin. The representation formulae given in Proposition 3.4 and Proposition 3.5 are equivalent.*

Proof. Suppose that the Stokes pair $(\vec{v}, p)$ is given by formulae (98), (99) in Proposition 3.5 with the three harmonic scalar functions Ψ, Φ and φ. Let ϕ be the solution of the equation

$$\Phi(\vec{x}) = \frac{4}{3}\phi(\vec{x}) + \frac{1}{3}\vec{x} \cdot \nabla\phi(\vec{x}). \tag{119}$$

We have $\phi(\vec{x}) = 3\int_0^1 \Phi(t\vec{x})\, dt$ (the proof for this is very similar to Lemma 2 in [29]) and $\Delta\phi(\vec{x}) = 0$ for $\vec{x} \in \Omega$. Now we define

$$\vec{h}(\vec{x}) = \frac{2}{3}\nabla\varphi(\vec{x}) + (\Psi(\vec{x}) + \vec{x} \cdot \nabla\Psi(\vec{x}))\,\vec{x} + \vec{x} \times \left(\frac{2}{3}\nabla\phi(\vec{x}) + \vec{x} \times \nabla\Psi(\vec{x})\right). \tag{120}$$

The structure of (120) implies

$$\vec{h}(\vec{x}) = \frac{2}{3}\nabla\varphi(\vec{x}) + \chi(\vec{x})\vec{x} + \vec{x} \times \vec{u}(\vec{x}),$$

where $\chi(\vec{x}) = \Psi(\vec{x}) + \vec{x} \cdot \nabla\Psi(\vec{x})$ and $\vec{u}(\vec{x}) = \frac{2}{3}\nabla\phi(\vec{x}) + \vec{x} \times \nabla\Psi(\vec{x})$. Further we have from (105) and (109)

$$\operatorname{rot}\vec{u}(\vec{x}) = -\nabla\chi(\vec{x}) \text{ and } \operatorname{div}\vec{u}(\vec{x}) = 0, \text{ for } \vec{x} \in \Omega,$$

which implies along with (104) and (107)

$$\Delta\vec{h}(\vec{x}) = \frac{2}{3}\nabla\Delta\varphi(\vec{x}) + \vec{x}\Delta\chi(\vec{x}) + 2\nabla\chi(\vec{x}) + \vec{x} \times \Delta\vec{u}(\vec{x}) + 2\operatorname{rot}\vec{u}(\vec{x}) = 0,$$

i.e. $\vec{h}$ is harmonic. By straightforward calculation from (120) there follows

$$-2\operatorname{div}\vec{h} = -6\chi - 4\vec{x} \cdot \nabla\chi = -6\Psi - 10\vec{x} \cdot \nabla\Psi - 4\vec{x} \cdot \nabla(\vec{x} \cdot \nabla\Psi),$$

which – by (99) – gives (95). Next we turn to the velocity formulae. From (120) follows

$$\vec{h}(\vec{x}) - \frac{1}{2}\nabla(\vec{x}\cdot\vec{h}(\vec{x})) - \frac{1}{2}\operatorname{rot}(\vec{x}\times\vec{h}(\vec{x})) = \vec{A}_{\varphi}(\vec{x}) + \vec{A}_{\phi}(\vec{x}) + \vec{A}_{\Psi}(\vec{x}),$$

where

$$\begin{aligned} 3\vec{A}_{\varphi}(\vec{x}) &= 2\nabla\varphi(\vec{x}) - \nabla(\vec{x}\cdot\nabla\varphi(\vec{x})) - \operatorname{rot}(\vec{x}\times\nabla\varphi(\vec{x})), \\ 3\vec{A}_{\phi}(\vec{x}) &= 2\vec{x}\times\nabla\phi(\vec{x}) - \operatorname{rot}(\vec{x}\times(\vec{x}\times\nabla\phi(\vec{x}))), \\ 2\vec{A}_{\Psi}(\vec{x}) &= \operatorname{rot}(r^2\vec{x}\times\nabla\Psi(\vec{x}) - \nabla(r^2(\Psi(\vec{x}) + \vec{x}\cdot\nabla\Psi(\vec{x}))) + \\ &\quad 2(\Psi(\vec{x}) + \vec{x}\cdot\nabla\Psi(\vec{x}))\vec{x} + 2\vec{x}\times(\vec{x}\times\nabla\Psi(\vec{x})). \end{aligned}$$

We simplify these expressions. First (109) and $\Delta\varphi(\vec{x}) = 0$ imply $\vec{A}_{\varphi}(\vec{x}) = \nabla\varphi(\vec{x})$. From (105), (110), (119) and $\Delta\phi(\vec{x}) = 0$ we have $\vec{A}_{\phi}(\vec{x}) = \vec{x}\times\nabla\Phi(\vec{x})$. From (110), (111), (112) follows

$$\vec{A}_{\Psi}(\vec{x}) = \operatorname{rot}(r^2\vec{x}\times\nabla\Psi(\vec{x})) - \frac{1}{2}(r^2\Delta\Psi(\vec{x}))\vec{x}.$$

Now $\Delta\Psi(\vec{x}) = 0$ implies $\vec{A}_{\Psi}(\vec{x}) = \operatorname{rot}(r^2\vec{x}\times\nabla\Psi(\vec{x}))$. We add the simplified expressions and use additionally the identity (101). There follows

$$\frac{3}{2}\vec{h}(\vec{x}) - \frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}(\vec{x})) + \vec{x}\times\operatorname{rot}\vec{h}(\vec{x})\right) = \nabla\varphi(\vec{x}) + \operatorname{rot}(r^2\vec{x}\times\nabla\Psi(\vec{x})) + \vec{x}\times\nabla\Phi(\vec{x}).$$

This proves (94) using (98).

Now suppose that the representation formulae of Proposition 3.4 hold. From $\Delta\vec{h}(\vec{x}) = 0$ follows that $\operatorname{rot}\vec{h}(\vec{x})$ and $-\operatorname{div}\vec{h}(\vec{x})$ are conjugate functions in the sense of Definition 3.6:

$$\operatorname{rot}\left(\operatorname{rot}\vec{h}(\vec{x})\right) = -\nabla\left(-\operatorname{div}\vec{h}(\vec{x})\right) \text{ and } \operatorname{div}\left(\operatorname{rot}\vec{h}(\vec{x})\right) = 0.$$

By Lemma 3.7 we have

$$\operatorname{rot}\vec{h}(\vec{x}) = \nabla\chi(\vec{x}) + \vec{x}\times\nabla\zeta(\vec{x}) \text{ and } -\operatorname{div}\vec{h}(\vec{x}) = \zeta(\vec{x}) + \vec{x}\cdot\nabla\zeta(\vec{x}) \qquad (121)$$

with harmonic scalar functions χ and ζ. Solving the equation

$$\zeta(\vec{x}) = -3\Psi(\vec{x}) - 2\vec{x}\cdot\nabla\Psi(\vec{x})$$

for the scalar function $\Psi(\vec{x})$ we obtain $\Psi(\vec{x}) = -\int_0^1 \frac{1}{2\tau^2} e^{\frac{3}{2}\left(1-\frac{1}{\tau}\right)}\zeta(\tau\vec{x})\,d\tau$ (similar calculation as in Lemma 3.7) and

$$\Delta\Psi(\vec{x}) = -\int_0^1 \frac{1}{2} e^{\frac{3}{2}\left(1-\frac{1}{\tau}\right)}(\Delta\zeta)(\tau\vec{x})\,d\tau = 0.$$

Substituting this into (121) and using $p = -2\operatorname{div}\vec{h}$ from Proposition 3.4 gives

$$p = -\left(6\Psi + 4\vec{x}\cdot\nabla\Psi\right) - \vec{x}\cdot\nabla\left(6\Psi + 4\vec{x}\cdot\nabla\Psi\right),$$

which is equivalent to formula (99) in Proposition 3.5. Using Proposition 3.4 we have $\operatorname{rot}\vec{v}(\vec{x}) = -4\nabla\phi(\vec{x}) + 2\operatorname{rot}\vec{h}(\vec{x})$, and from (121)

$$\operatorname{rot}\vec{v}(\vec{x}) = -\nabla\left(4\phi(\vec{x}) - 2\chi(\vec{x})\right) - \vec{x}\times\nabla\left(6\Psi(\vec{x}) + 4\vec{x}\cdot\nabla\Psi(\vec{x})\right). \tag{122}$$

The solution of the equation

$$4\phi\left(\vec{x}\right) - 2\chi\left(\vec{x}\right) = \Phi\left(\vec{x}\right) + \vec{x}\cdot\nabla\Phi\left(\vec{x}\right)$$

is (by a similar calculation as in Lemma 3.7 again)

$$\Phi\left(\vec{x}\right) = \int_0^1 \frac{1}{\tau^2} e^{1-\frac{1}{\tau}}\left(4\phi\left(\tau\vec{x}\right) - 2\chi\left(\tau\vec{x}\right)\right)d\tau,$$

for which we have $\Delta\Phi\left(\vec{x}\right) = 0$. Substituting this into (122) there follows

$$\operatorname{rot}\vec{v}(\vec{x}) = -\nabla\left(\Phi(\vec{x}) + \vec{x}\cdot\nabla\Phi(\vec{x})\right) - \vec{x}\times\nabla\left(6\Psi(\vec{x}) + 4\vec{x}\cdot\nabla\Psi(\vec{x})\right)$$

and by a calculation using (111), (112) we have

$$\begin{aligned}\operatorname{rot}\left(\vec{v}(\vec{x}) - \operatorname{rot}\left(r^2\vec{x}\times\nabla\Psi(\vec{x})\right) - \vec{x}\times\nabla\Phi(\vec{x})\right) &= 0,\\ \operatorname{div}\left(\vec{v}(\vec{x}) - \operatorname{rot}\left(r^2\vec{x}\times\nabla\Psi(\vec{x})\right) - \vec{x}\times\nabla\Phi(\vec{x})\right) &= 0.\end{aligned}$$

The latter equation holds trivially. Again by Lemma 3.7 there follows

$$\begin{aligned}\vec{v}(\vec{x}) - \operatorname{rot}\left(r^2\vec{x}\times\nabla\Psi(\vec{x})\right) - \vec{x}\times\nabla\Phi(\vec{x}) &= \nabla\varphi(\vec{x}) + \vec{x}\times\nabla\omega(\vec{x}),\\ 0 &= \omega(\vec{x}) + \vec{x}\cdot\nabla\omega(\vec{x})\end{aligned}$$

with harmonic scalar functions φ and ω. Moreover, the latter equation shows that $\omega \equiv 0$ in the domain (see the last part of the proof of Lemma 3.7). So we have for $\vec{v}$

$$\vec{v}(\vec{x}) = \nabla\varphi(\vec{x}) + \operatorname{rot}\left(r^2\vec{x}\times\nabla\Psi(\vec{x})\right) + \vec{x}\times\nabla\Phi(\vec{x}).$$

This is formula (98) in Proposition 3.5.

So choosing the functions appropriately in Proposition 3.4 and in Proposition 3.5 they give equivalent formulae for the Stokes pair in a star-shaped domain. □

Remark 3.9 This result is mainly based on the following formulae computed from (98) and (99).

$$\begin{aligned}\operatorname{rot}\vec{v} &= -\nabla\left(\Phi + \vec{x}\cdot\nabla\Phi\right) - \vec{x}\times\nabla\left(6\Psi + 4\vec{x}\cdot\nabla\Psi\right)\\ p &= -\left(6\Psi + 4\vec{x}\cdot\nabla\Psi\right) - \vec{x}\cdot\nabla\left(6\Psi + 4\vec{x}\cdot\nabla\Psi\right)\end{aligned}$$

They show that $\operatorname{rot}\vec{v}$ and p are conjugate (harmonic) functions in a domain if $\vec{v}$ is a Stokes function there with the corresponding pressure p. (And vice versa: if $\vec{q}$ is conjugate to p, then any divergence free vector potential function of $\vec{q}$ is a Stokes function with the corresponding pressure p.) □

Remark 3.10 In [38] there is given a representation – called Papkovich-Neuber representation – of Stokes functions $\vec{v}$ and the corresponding pressures p in terms of the harmonic potentials $\vec{H} = \vec{H}(\vec{x})$ and $H_0 = H_0(\vec{x})$:

$$\vec{v}(\vec{x}) = -\nabla H_0(\vec{x}) - \nabla\left(\vec{x}\cdot\vec{H}(\vec{x})\right) + 2\vec{H}(\vec{x}), \tag{123}$$

$$p(\vec{x}) = -2\operatorname{div}\vec{H}(\vec{x}). \tag{124}$$

If the domain is star-shaped, then from (123) follows

$$\operatorname{rot}\vec{v} = 2\operatorname{rot}\vec{H},$$

which we substitute into (97):

$$\phi(\vec{x}) = -\frac{1}{2}\int_0^1 t^4\vec{x}\cdot\operatorname{rot}\vec{H}(t\vec{x})dt,$$

and into (96):

$$\vec{h} = -\frac{2}{3}\nabla H_0 + \frac{5}{3}\vec{H}(\vec{x}) - \frac{1}{3}\nabla\left(\vec{x}\cdot\vec{H}(\vec{x})\right) + \frac{1}{3}\operatorname{rot}\left(\vec{x}\times\vec{H}(\vec{x})\right) + \frac{2}{3}\vec{x}\times\nabla\phi(\vec{x}).$$

These formulae constitute a connection between the auxiliary harmonic function $\vec{h}$ in Proposition 3.4 and the Papkovich-Neuber potentials. Moreover we have

$$\operatorname{rot}(\vec{H}-\vec{h}) = -\nabla(2\phi), \quad \operatorname{div}(\vec{H}-\vec{h}) = 0.$$

□

Remark 3.11 There is, of course, also a connection between the auxiliary harmonic functions φ, Ψ and Φ in Proposition 3.5 and the harmonic potentials H_0 and $\vec{H}$ in (123) and (124) if the domain is star-shaped. If we set

$$\begin{aligned}\vec{H}(\vec{x}) &:= \left(\Psi(\vec{x}) + 2\vec{x}\cdot\nabla\Psi(\vec{x})\right)\vec{x} - r^2\nabla\Psi(\vec{x}) + \frac{1}{2}\vec{x}\times\nabla\Phi(\vec{x}),\\ H_0(\vec{x}) &:= -\varphi(\vec{x}),\end{aligned}$$

where $r = |\vec{x}|$, then by (109)–(112) the equalities

$$\begin{aligned}
-\nabla\left(\vec{x}\cdot\vec{H}(\vec{x})\right) + 2\vec{H}(\vec{x}) &= \mathrm{rot}\left(r^2\vec{x}\times\nabla\Psi(\vec{x})\right) + \vec{x}\times\nabla\Phi(\vec{x}),\\
-2\,\mathrm{div}\,\vec{H}(\vec{x}) &= -6\Psi(\vec{x}) - 10\vec{x}\cdot\nabla\Psi(\vec{x}) - 4\vec{x}\cdot\nabla\left(\vec{x}\cdot\nabla\Psi(\vec{x})\right)
\end{aligned}$$

are easily verified and constitute the connection between (98),(99) and (123),(124). By (119) there also follows the connection

$$\vec{H}(\vec{x}) + \frac{2}{3}\nabla H_0(\vec{x}) = \vec{h}(\vec{x}) + \frac{1}{6}\vec{x}\times\nabla\left(\vec{x}\cdot\nabla\phi(\vec{x})\right)$$

between the harmonic functions from the representations (123),(124) and Proposition 3.4. □

3.1.2 Representation in arbitrary spatial domains

In this subsection we study the question how to avoid the restriction of star-shapedness for the domain. This could be done – as pointed out in [29] – by proving the solvability of a certain equation in the domain (precisely the equation (20) in [29]).

To avoid this we can choose another way of this generalization with an additional complement term in the velocity formula in Proposition 3.4. Moreover, instead of one auxiliary harmonic function, we use a harmonic vector function and also a harmonic scalar function (as in the case of the Papkovich-Neuber potentials).

Theorem 3.12 *Let $\Omega \subseteq \mathbb{R}^3$ be a domain. $(\vec{v}, p)$ is a Stokes pair on Ω iff there are harmonic functions $\vec{w}$ and ψ on Ω such that*

$$\begin{aligned}
\vec{v} &= -\frac{1}{2}\nabla\left(\vec{x}\cdot\vec{w}\right) - \frac{1}{2}\,rot\left(\vec{x}\times\vec{w} + \psi\vec{x}\right) + \vec{w}, &(125)\\
p &= -2\,div\,\vec{w}. &(126)
\end{aligned}$$

Moreover, these harmonic functions are

$$\begin{aligned}
\vec{w} &= \frac{2}{3}\vec{v} - \frac{1}{6}\left(p\vec{x} - \vec{x}\times rot\,\vec{v}\right), &(127)\\
\psi &= -\frac{1}{6}\vec{x}\cdot rot\,\vec{v}. &(128)
\end{aligned}$$

Proof. The proof is very similar to that of Theorem 2 in [29]. Let $\vec{v}$ and p be given by (125) and (126) with harmonic $\vec{w}$ and ψ. Using equations (102)–(107) we have

$$\Delta\left(-\frac{1}{2}\nabla\left(\vec{x}\cdot\vec{w}\right) + \vec{w}\right) = \frac{1}{2}p, \quad \Delta\left(-\frac{1}{2}\,\mathrm{rot}\left(\vec{x}\times\vec{w} + \psi\vec{x}\right)\right) = \frac{1}{2}p,$$

and

$$\operatorname{div}\vec{v} = \operatorname{div}\left(-\frac{1}{2}\nabla\left(\vec{x}\cdot\vec{w}\right) + \vec{w}\right) - \frac{1}{2}\Delta\left(\vec{x}\cdot\vec{w}\right) + \operatorname{div}\vec{w} = 0,$$

so $(\vec{v}, p)$ is a Stokes pair.

Let now $(\vec{v}, p)$ be a Stokes pair. By (90) we have $\Delta\vec{v} = -\operatorname{rot}\operatorname{rot}\vec{v} = \nabla p$, $\operatorname{rot}\Delta\vec{v} = 0$ and $\Delta p = 0$. Using (102)–(107) we calculate (see also (127)):

$$\begin{aligned}\Delta\left[\frac{2}{3}\vec{v} - \frac{1}{6}\left(p\vec{x} - \vec{x}\times\operatorname{rot}\vec{v}\right)\right] &= \frac{2}{3}\Delta\vec{v} - \frac{1}{6}(2\nabla p + \vec{x}\Delta p - \vec{x}\times\operatorname{rot}\Delta\vec{v} \\ &\quad -2\operatorname{rot}\operatorname{rot}\vec{v}) \\ &= \frac{2}{3}\Delta\vec{v} - \frac{1}{6}(2\Delta\vec{v} + 2\Delta\vec{v}) = 0.\end{aligned}$$

We have further:

$$\begin{aligned}\operatorname{div}\left[\frac{2}{3}\vec{v} - \frac{1}{6}\left(p\vec{x} - \vec{x}\times\operatorname{rot}\vec{v}\right)\right] &= -\frac{1}{6}\left(3p + \vec{x}\cdot\nabla p + \vec{x}\cdot\operatorname{rot}\operatorname{rot}\vec{v}\right) \\ &= -\frac{1}{6}\left(3p + \vec{x}\cdot\Delta\vec{v} - \vec{x}\cdot\Delta\vec{v}\right) = -\frac{1}{2}p,\end{aligned}$$

which gives (126) with harmonic $\vec{w}$ defined by (127). Again by (102)–(107) we obtain

$$\begin{aligned}\nabla(\vec{x}\cdot\vec{w}) &= \frac{2}{3}\nabla(\vec{x}\cdot\vec{v}) - \frac{1}{3}p\vec{x} - \frac{1}{6}r^2\nabla p, \\ \operatorname{rot}(\vec{x}\times\vec{w}) &= \frac{2}{3}\operatorname{rot}(\vec{x}\times\vec{v}) - \frac{1}{6}\vec{x}\times\nabla(\vec{x}\cdot\operatorname{rot}\vec{v}) - \frac{1}{3}\vec{x}\times\operatorname{rot}\vec{v} + \frac{1}{6}r^2\nabla p,\end{aligned}$$

where $r = |\vec{x}|$. By (101) we compose

$$-\frac{1}{2}\nabla\left(\vec{x}\cdot\vec{w}\right) - \frac{1}{2}\operatorname{rot}\left(\vec{x}\times\vec{w}\right) + \vec{w} = \vec{v} + \frac{1}{12}\vec{x}\times\nabla(\vec{x}\cdot\operatorname{rot}\vec{v}),$$

which implies the representation formula (125) using the definition (128). □

Remark 3.13 The representation formulae (125), (126) in Theorem 3.12 are also comparable to the Papkovich-Neuber representation (123), (124):

$$\begin{aligned}\vec{w} &= -\frac{2}{3}\nabla H_0 - \frac{1}{3}\nabla(\vec{x}\cdot\vec{H}) + \frac{1}{3}\operatorname{rot}(\vec{x}\times\vec{H}) + \frac{5}{3}\vec{H}, \\ \psi &= -\frac{1}{3}\vec{x}\cdot\operatorname{rot}\vec{H}.\end{aligned}$$

Similar to the connections between the harmonic functions $\vec{H}, \vec{h}$ and ϕ in Remark 3.10 we have here also

$$\operatorname{rot}(\vec{H} - \vec{w}) = -\nabla(2\psi), \quad \operatorname{div}(\vec{H} - \vec{w}) = 0.$$

□

Remark 3.14 The auxiliary harmonic functions in Theorem 3.12 are not uniquely determined by the Stokes function $\vec{v}$ and its corresponding pressure p. Assume that for some harmonic $\vec{w}$ and ψ

$$-\frac{1}{2}\nabla(\vec{x}\cdot\vec{w}) - \frac{1}{2}\operatorname{rot}(\vec{x}\times\vec{w}+\psi\vec{x}) + \vec{w} = 0$$

in a spatial domain Ω (see (125)). If we take the rotation of this equation, then we obtain by (109) and $\Delta\psi = 0$ the equation

$$\operatorname{rot}\left(\vec{w} - \frac{1}{2}\operatorname{rot}(\vec{x}\times\vec{w})\right) = \frac{1}{2}\nabla(\psi + \vec{x}\cdot\nabla\psi).$$

We have $\nabla\operatorname{div}\vec{w} = 0$ by (126), which gives $\operatorname{div}\vec{w} = 3\gamma \in \mathbb{R}$ in Ω. This implies $\operatorname{div}\vec{w}_0 = 0$ for $\vec{w}_0 := \vec{w} - \gamma\vec{x}$, from which there follows $\vec{x}\times\vec{w} = \vec{x}\times\vec{w}_0$ and by $\operatorname{rot}\vec{x} = 0$ also

$$\begin{aligned}\operatorname{rot}(2\vec{w}_0 - \operatorname{rot}(\vec{x}\times\vec{w}_0)) &= \nabla(\psi + \vec{x}\cdot\nabla\psi),\\ \operatorname{div}(2\vec{w}_0 - \operatorname{rot}(\vec{x}\times\vec{w}_0)) &= 0.\end{aligned}$$

These equations are also valid for $\tilde{\vec{w}}_0 := \vec{w}_0 + \nabla\omega$ and $\tilde{\psi} := \psi + C$ with arbitrary ω harmonic in Ω and $C \in \mathbb{R}$. This means that the harmonic functions in Theorem 3.12 are certainly not uniquely determined by the Stokes function and its corresponding pressure. (The above equations show that in fact the functions $2\vec{w}_0 - \operatorname{rot}(\vec{x}\times\vec{w}_0)$ and $-\psi - \vec{x}\cdot\nabla\psi$ are conjugate in the sense of Definition 3.6.) □

3.2 Linear elasticity

In this section we investigate representations of the solutions $\vec{v}$ and p of the equations

$$\Delta\vec{v} = \nabla p, \quad \operatorname{div}\vec{v} = \nu p \tag{129}$$

in domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. While in the two-dimensional case there is no restriction for the domain Ω, in the three-dimensional case Ω is assumed to be star-shaped. (We obtain for $\nu = 0$, of course, Stokes pairs, see (90).)

Remark 3.15 Consider Navier's equation for the linear elasticity problem

$$(\lambda + \mu)\nabla\operatorname{div}\vec{v} + \mu\Delta\vec{v} = 0 \tag{130}$$

for the (small) displacement vector $\vec{v} = \vec{v}(\vec{x}) \in \mathbb{R}^n$, $n = 2, 3$, of a body of isotropic material, where λ and μ are the Lamé constants. Introduce

$$p := -\frac{\lambda+\mu}{\mu}\operatorname{div}\vec{v}, \tag{131}$$

which is (in the incompressible limit) the hydrostatic pressure divided by the Poisson ratio $\frac{\lambda}{2(\lambda+\mu)}$. By setting

$$\nu := -\frac{\mu}{\lambda+\mu}$$

we have instead of (130) the equations (129). In a general (two or three-dimensional) domain the solution of (130) is representable – similar to (123) and (124) – by the (not unique) harmonic functions $H_0 = H_0(\vec{x})$ and $\vec{H} = \vec{H}(\vec{x})$:

$$\vec{v} = 2\vec{H} - \frac{\lambda+\mu}{\lambda+2\mu}\nabla(H_0 + \vec{x}\cdot\vec{H}) = 2\vec{H} - \frac{1}{1-\nu}\nabla(H_0 + \vec{x}\cdot\vec{H}),$$

if $\nu \neq 1$ (or equivalently $\lambda + 2\mu \neq 0$). □

Remark 3.16 The equations (129) could be considered also as an eigenvalue problem, if some boundary conditions (e.g. homogeneous Dirichlet) for $\vec{v}$ are given. □

If the problem domain is specified to be star-shaped, then we obtain the following representation, which constitutes a generalization of Proposition 3.4:

Theorem 3.17 *Let $\Omega \subset \mathbb{R}^3$ be a star-shaped domain with respect to the origin, and set $\nu \in \mathbb{R}$, $\nu \neq \frac{3}{4}, 1$. The functions $\vec{v} \in C_2(\Omega)$ and $p \in C_1(\Omega)$ satisfy* (129) *iff there exists a harmonic function $\vec{h} \in C_2(\Omega)$ such that*

$$\vec{v}(\vec{x}) = -\frac{1}{2}\left(\vec{x}\,\mathrm{div}\,\vec{h}(\vec{x}) + \vec{x}\times\mathrm{rot}\,\vec{h}(\vec{x})\right) + \left(\frac{3}{2} - 2\nu\right)\vec{h}(\vec{x}), \quad (132)$$

$$p(\vec{x}) = -2\,\mathrm{div}\,\vec{h}(\vec{x}), \text{ for } \vec{x}\in\Omega. \quad (133)$$

The harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}(\vec{x}) = \frac{2}{3-4\nu}\left(\vec{v}(\vec{x}) - \frac{1}{4}\left(p(\vec{x})\vec{x} - \frac{1}{1-\nu}\vec{x}\times\mathrm{rot}\,\vec{v}(\vec{x})\right) + \vec{x}\times\nabla\phi(\vec{x})\right), \quad (134)$$

where the function ϕ is harmonic in Ω and defined by

$$\phi(\vec{x}) = -\frac{1}{4(1-\nu)}\int_0^1 t^{4(1-\nu)}\vec{x}\cdot\mathrm{rot}\,\vec{v}(t\vec{x})dt. \quad (135)$$

Proof. First assume that $\vec{v}$ and p are given by (132) and (133) with $\vec{h}$ harmonic in Ω. Using the identities (102)–(106), and $\Delta\vec{h} = \nabla \operatorname{div} \vec{h} - \operatorname{rot}\operatorname{rot}\vec{h}$ there follows

$$\begin{aligned}
\Delta\vec{v} &= -\frac{1}{2}\left(2\nabla \operatorname{div}\vec{h} + \vec{x}\cdot \operatorname{div}\Delta\vec{h} + \vec{x}\times \operatorname{rot}\Delta\vec{h} + 2\operatorname{rot}\operatorname{rot}\vec{h}\right) \\
&= -2\nabla \operatorname{div}\vec{h} = \nabla p, \\
\operatorname{div}\vec{v} &= -\frac{1}{2}\left(3\operatorname{div}\vec{h} + \vec{x}\cdot\nabla \operatorname{div}\vec{h} - \vec{x}\cdot \operatorname{rot}\operatorname{rot}\vec{h}\right) + \left(\frac{3}{2} - 2\nu\right)\operatorname{div}\vec{h} \\
&= -\frac{1}{2}\left(3\operatorname{div}\vec{h} + \vec{x}\cdot\Delta\vec{h}\right) + \left(\frac{3}{2} - 2\nu\right)\operatorname{div}\vec{h} = -2\nu \operatorname{div}\vec{h} = \nu p,
\end{aligned}$$

so $\vec{v}$ and p fulfil (129).

Next, let $\vec{v}$ and p satisfy (129). There follows

$$\operatorname{rot}\Delta\vec{v} = 0 \text{ and } (\nu - 1)\Delta p = 0,$$

and $\Delta p = 0$, because $\nu \neq 1$. Let $\vec{h}$ and ϕ be defined by (134) and (135). The function $k(\vec{x}) := \vec{x}\cdot \operatorname{rot}\vec{v}(\vec{x})$ by (103) is harmonic in Ω. We have further

$$\Delta\phi(\vec{x}) = \frac{1}{4(1-\nu)}\Delta\left(\int_0^1 t^{3-4\nu}k(t\vec{x})dt\right) = \frac{1}{4(1-\nu)}\int_0^1 t^{5-4\nu}(\Delta k)(t\vec{x})dt = 0,$$

which means that ϕ is harmonic in Ω. By partial integration we obtain

$$\begin{aligned}
\vec{x}\cdot\nabla\phi(\vec{x}) &= -\frac{1}{4(1-\nu)}\vec{x}\cdot\int_0^1 t^{4-4\nu}(\nabla k)(t\vec{x})dt \\
&= -\frac{1}{4(1-\nu)}\int_0^1 t^{4-4\nu}\frac{d}{dt}(k(t\vec{x}))dt \\
&= -\frac{1}{4(1-\nu)}k(\vec{x}) + \int_0^1 t^{3-4\nu}k(t\vec{x})dt \\
&= -\frac{1}{4(1-\nu)}\vec{x}\cdot \operatorname{rot}\vec{v}(\vec{x}) - 4(1-\nu)\phi(\vec{x}),
\end{aligned}$$

which gives that the function ϕ satisfies the equation

$$4(1-\nu)\phi(\vec{x}) + \vec{x}\cdot\nabla\phi(\vec{x}) + \frac{1}{4(1-\nu)}\vec{x}\cdot \operatorname{rot}\vec{v}(\vec{x}) = 0. \tag{136}$$

Using again the identities (102)–(109) and (129) we calculate from (134):

$$\begin{aligned}
\Delta\vec{h} &= \frac{2}{3-4\nu}\left(\Delta\vec{v} - \frac{1}{4}\left(2\nabla p - \frac{2}{1-\nu}\operatorname{rot}\operatorname{rot}\vec{v}\right)\right) \\
&= \frac{2}{3-4\nu}\left(\nabla p - \frac{1}{4}\left(2\nabla p - \frac{2}{1-\nu}(\nu\nabla p - \nabla p)\right)\right) = 0, \\
\operatorname{div}\vec{h} &= \frac{2}{3-4\nu}\left(\operatorname{div}\vec{v} - \frac{1}{4}\left(3p + \vec{x}\cdot\nabla p + \frac{1}{1-\nu}\vec{x}\cdot\operatorname{rot}\operatorname{rot}\vec{v}\right)\right) = \\
&= \frac{2}{3-4\nu}\left(\nu p - \frac{1}{4}\left(3p + \vec{x}\cdot\nabla p + \frac{1}{1-\nu}\vec{x}\cdot(\nu\nabla p - \nabla p)\right)\right) = \\
&= \frac{2}{3-4\nu}\left(\nu p - \frac{3}{4}p\right) = -\frac{1}{2}p,
\end{aligned}$$

which gives (133). With the help of (105), (129) and the identity

$$\operatorname{rot}(\vec{x}\times\operatorname{rot}\vec{v}) = \vec{x}\times\nabla\operatorname{div}\vec{v} - \nabla(\vec{x}\cdot\operatorname{rot}\vec{v}) - \operatorname{rot}\vec{v} - \vec{x}\times\Delta\vec{v}$$

we obtain

$$\begin{aligned}
\operatorname{rot}\left(p\vec{x} - \frac{1}{1-\nu}\vec{x}\times\operatorname{rot}\vec{v}\right) &= -\vec{x}\times\nabla p - \frac{1}{1-\nu}\left((\nu-1)\vec{x}\times\nabla p\right. \\
&\quad \left.-\nabla(\vec{x}\cdot\operatorname{rot}\vec{v}) - \operatorname{rot}\vec{v}\right) \\
&= \frac{1}{1-\nu}\left(\nabla(\vec{x}\cdot\operatorname{rot}\vec{v}) + \operatorname{rot}\vec{v}\right),
\end{aligned}$$

and using (109) along with $\Delta\phi = 0$ and (136) there follows

$$\begin{aligned}
\operatorname{rot}\vec{h} &= \frac{1}{2(1-\nu)}\operatorname{rot}\vec{v} - \frac{2}{3-4\nu}\nabla\left(\phi + \vec{x}\cdot\nabla\phi + \frac{1}{4(1-\nu)}\vec{x}\cdot\operatorname{rot}\vec{v}\right) \\
&= -2\nabla\phi + \frac{1}{2(1-\nu)}\operatorname{rot}\vec{v}.
\end{aligned}$$

We combine these equalities

$$\begin{aligned}
\vec{v}(\vec{x}) &= -\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}(\vec{x}) + \vec{x}\times\operatorname{rot}\vec{h}(\vec{x})\right) + \left(\frac{3}{2} - 2\nu\right)\vec{h}(\vec{x}) - \\
&\quad \frac{1}{3-4\nu}\vec{x}\times\left(4(1-\nu)\phi(\vec{x}) + \vec{x}\cdot\nabla\phi(\vec{x}) + \frac{1}{4(1-\nu)}\vec{x}\cdot\operatorname{rot}\vec{v}(\vec{x})\right),
\end{aligned}$$

which gives (132) using (136).

It remains to show the uniqueness of the harmonic function $\vec{h}$. Suppose the representation (132) is valid for harmonic $\vec{h}_1$ and $\vec{h}_2$, too. Then we have

$$-\frac{1}{2}\left(\vec{x}\operatorname{div}(\vec{h}_1 - \vec{h}_2) + \vec{x}\times\operatorname{rot}(\vec{h}_1 - \vec{h}_2)\right) + \left(\frac{3}{2} - 2\nu\right)(\vec{h}_1 - \vec{h}_2) = 0$$

If we set $\vec{h}_0 := \vec{h}_1 - \vec{h}_2$, then

$$-\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}_0 + \vec{x}\times\operatorname{rot}\vec{h}_0\right) + \left(\frac{3}{2} - 2\nu\right)\vec{h}_0 = 0 \tag{137}$$

follows, from which we obtain

$$\nu\operatorname{div}\vec{h}_0(\vec{x}) = 0$$

for $\vec{x}\in\Omega$ by taking the divergence.

If $\nu = 0$ (in fact this is the case of Stokes functions), then we have $\vec{h}_0 = \gamma\vec{x}$ for some $\gamma\in\mathbb{R}$ as in the proof of Proposition 4 in [29]. (The function p in (129) is determined for $\nu = 0$ up to an additive constant. This constant is in fact γ.)

If $\nu \neq 0$, then $\operatorname{div}\vec{h}_0 = 0$ in Ω, which we substitute into (137):

$$-\frac{1}{2}\vec{x}\times\operatorname{rot}\vec{h}_0 + \left(\frac{3}{2} - 2\nu\right)\vec{h}_0 = 0. \tag{138}$$

The rotation of (138) gives

$$4(1-\nu)\operatorname{rot}\vec{h}_0 + \nabla\left(\vec{x}\cdot\operatorname{rot}\vec{h}_0\right) = 0. \tag{139}$$

Multiply this by $\vec{x}$ and set $\psi(\vec{x}) := \vec{x}\cdot\operatorname{rot}\vec{h}_0(\vec{x})$. We have $\Delta\psi = 0$ by (103), $\nabla\psi(\vec{x}) = -4(1-\nu)\operatorname{rot}\vec{h}_0$ by (138), and

$$4(1-\nu)\psi(\vec{x}) + \vec{x}\cdot\nabla\psi(\vec{x}) = 0. \tag{140}$$

Similar to the proof of Proposition 4 in [29], we have for a fixed $\varepsilon > 0$ a neighbourhood of the origin $U_\varepsilon(0)\subseteq\Omega$, because $0\in\Omega$. Set $f(t) := \psi(t\vec{x})$ for a fixed $\vec{x}\in U_\varepsilon(0)$. There follows $f(0) = 0$ and by (140) we also have

$$4(1-\nu)f(t) + tf'(t) = 0$$

in the interval $[0,1]$. Hence

$$f(t) = \begin{cases} 0, & \text{for } \nu < 1, \\ Ct^{-4(1-\nu)}, & \text{for } \nu > 1, \text{ where } C\in\mathbb{R}, \end{cases}$$

from which follows

$$\psi(\vec{x}) = f(1) = \begin{cases} 0, & \text{for } \nu < 1, \\ C, & \text{for } \nu > 1, \end{cases}$$

for $\vec{x} \in U_\varepsilon(0)$. Substitute this into (139):

$$4(1-\nu)\,\mathrm{rot}\,\vec{h}_0 = 0,$$

which implies $\mathrm{rot}\,\vec{h}_0 = 0$ in $U_\varepsilon(0)$ because $\nu \neq 1$. Now (138) implies

$$\left(\frac{3}{2} - 2\nu\right)\vec{h}_0 = 0,$$

and by $\nu \neq \frac{3}{4}$ we also have $\vec{h}_0 = 0$ in $U_\varepsilon(0)$. Since $\vec{h}_0$ is harmonic in Ω, there follows $\vec{h}_0 = 0$, and hence also $\vec{h}_1 - \vec{h}_2 = 0$, identically in Ω. This completes the proof. □

Remark 3.18 The Lamé constants in the linear elasticity problem (130) satisfy usually $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$ (the latter quantity is called compression modulus), that is $\lambda > -\frac{2}{3}\mu$. The restrictions $\nu \neq \frac{3}{4}$ and $\nu \neq 1$ in Theorem 3.17 mean for the Lamé constants $\lambda \neq -\frac{7}{3}\mu$ and $\lambda \neq -2\mu$, respectively. Hence we see, that Theorem 3.17 could be used to represent the solution of (130) by an auxiliary harmonic vector function in a three-dimensional star-shaped domain for arbitrary allowed values (i.e. satisfying $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$) of the Lamé constants. □

Remark 3.19 If we use $\tilde{\vec{h}} := \frac{2}{3-4\nu}\vec{h}$ instead of $\vec{h}$ of Theorem 3.17, then the representation formulae (132) and (133) are altered to

$$\begin{aligned} \vec{v}(\vec{x}) &= -\frac{1}{3-4\nu}\left(\vec{x}\,\mathrm{div}\,\tilde{\vec{h}}(\vec{x}) + \vec{x} \times \mathrm{rot}\,\tilde{\vec{h}}(\vec{x})\right) + \tilde{\vec{h}}(\vec{x}), \\ p(\vec{x}) &= -\frac{4}{3-4\nu}\,\mathrm{div}\,\tilde{\vec{h}}(\vec{x}). \end{aligned}$$

Setting $\nu = 0$ into these equations we get, of course, the case of Stokes functions and exactly their representation given by Theorem 2 in [29]. □

To complete the generalization of the representation theorems we consider the two-dimensional counterpart of Theorem 3.17.

Theorem 3.20 *Let $\Omega \subseteq \mathbb{R}^2$ be a domain and set $\nu \in \mathbb{R}$, $\nu \neq \frac{1}{2}, 1$. The functions $\vec{v} \in C_2(\Omega)$ and $p \in C_1(\Omega)$ satisfy (129) iff there exists a harmonic function $\vec{h} \in C_2(\Omega)$ such that*

$$\vec{v}\,(\vec{x}) = -\frac{1}{2}\left(\vec{x}\,\mathrm{div}\,\vec{h}\,(\vec{x}) + \vec{x}^{\perp}\,\mathrm{rot}\,\vec{h}\,(\vec{x})\right) + (1-2\nu)\vec{h}\,(\vec{x}) \qquad (141)$$

$$p\,(\vec{x}) = -2\,\mathrm{div}\,\vec{h}\,(\vec{x}) \;\; \textit{for } \vec{x} \in \Omega; \qquad (142)$$

this harmonic function $\vec{h}$ is unique, and we have

$$\vec{h}\,(\vec{x}) = \frac{1}{1-2\nu}\left(\vec{v}\,(\vec{x}) - \frac{1}{4}\left(p\,(\vec{x})\,\vec{x} - \frac{1}{1-\nu}\vec{x}^{\perp}\,rot\,\vec{v}\,(\vec{x})\right)\right) \;\; for\ \vec{x} \in \Omega. \tag{143}$$

Proof. We use the same identities which are also used in [29]:

$$\begin{aligned}
\operatorname{div}(\varphi\vec{x}) &= 2\varphi + \vec{x}\cdot\nabla\varphi && (144)\\
\operatorname{div}(\vec{x}^{\perp}\operatorname{rot}\vec{v}) &= \vec{x}\cdot(\Delta\vec{v} - \nabla\operatorname{div}\vec{v}) && (145)\\
\operatorname{rot}(\varphi\vec{x}) &= \vec{x}^{\perp}\cdot\nabla\varphi && (146)\\
\operatorname{rot}(\vec{x}^{\perp}\operatorname{rot}\vec{v}) &= \vec{x}^{\perp}\cdot(\Delta\vec{v} - \nabla\operatorname{div}\vec{v}) - 2\operatorname{rot}\vec{v} && (147)\\
\Delta(\varphi\vec{x}) &= 2\nabla\varphi + \vec{x}\Delta\varphi && (148)\\
\Delta(\vec{x}^{\perp}\operatorname{rot}\vec{v}) &= \vec{x}^{\perp}\operatorname{rot}(\Delta\vec{v}) - 2(\Delta\vec{v} - \nabla\operatorname{div}\vec{v}) && (149)
\end{aligned}$$

First, assume $\vec{v}$ and p are given by (141) and (142) where $\vec{h}$ is harmonic. Using (148) and (149) we calculate

$$\begin{aligned}
\Delta\vec{v} &= -\frac{1}{2}\left(2\nabla\operatorname{div}\vec{h} + \vec{x}\operatorname{div}\Delta\vec{h} + \vec{x}^{\perp}\operatorname{rot}\Delta\vec{h} - 2\Delta\vec{h} + 2\nabla\operatorname{div}\vec{h}\right)\\
&\quad +(1-2\nu)\Delta\vec{h}\\
&= -2\nabla\operatorname{div}\vec{h} = \nabla p,
\end{aligned}$$

and by (144) and (145) we obtain for the divergence

$$\begin{aligned}
\operatorname{div}\vec{v} &= -\frac{1}{2}\left(2\operatorname{div}\vec{h} + \vec{x}\cdot\nabla\operatorname{div}\vec{h} + \vec{x}\cdot\Delta\vec{h} - \vec{x}\cdot\nabla\operatorname{div}\vec{h}\right) + (1-2\nu)\operatorname{div}\vec{h}\\
&= -2\nu\operatorname{div}\vec{h} = \nu p,
\end{aligned}$$

hence $\vec{v}$ and p fulfil (129).

Next, suppose that $\vec{v}$ and p satisfy (129). There follows $\operatorname{rot}\Delta\vec{v} = 0$. We have further $(\nu-1)\Delta p = 0$ and therefore $\Delta p = 0$ because $\nu \neq 1$. Let $\vec{h}$ be the function given by (143). We calculate by (148) and (149):

$$\begin{aligned}
\Delta\vec{h} &= \frac{1}{1-2\nu}\left[\Delta\vec{v} - \frac{1}{4}\left(2\nabla p + \vec{x}\Delta p - \frac{1}{1-\nu}\left(\vec{x}^{\perp}\operatorname{rot}\Delta\vec{v} - 2\Delta\vec{v} + 2\nabla\operatorname{div}\vec{v}\right)\right)\right]\\
&= \frac{1}{1-2\nu}\left[\nabla p - \frac{1}{4}\left(2\nabla p - \frac{1}{1-\nu}\left(2\nu\nabla p - 2\nabla p\right)\right)\right] = 0,
\end{aligned}$$

and moreover we obtain by (144) and (145)

$$\begin{aligned}
\operatorname{div}\vec{h} &= \frac{1}{1-2\nu}\left[\operatorname{div}\vec{v}-\frac{1}{4}\left(2p+\vec{x}\cdot\nabla p-\frac{1}{1-\nu}\left(\vec{x}\cdot\Delta\vec{v}-\vec{x}\cdot\nabla\operatorname{div}\vec{v}\right)\right)\right]\\
&= \frac{1}{1-2\nu}\left[\nu p-\frac{1}{4}\left(2p+\vec{x}\cdot\nabla p-\frac{1}{1-\nu}\left(\vec{x}\cdot\nabla p-\nu\vec{x}\cdot\nabla p\right)\right)\right]\\
&= \frac{1}{1-2\nu}\left[\nu-\frac{1}{2}\right]p=-\frac{1}{2}p,
\end{aligned}$$

which gives (142) with harmonic $\vec{h}$. By (146) and (147) we have

$$\begin{aligned}
\operatorname{rot}\vec{h} &= \frac{1}{1-2\nu}\left[\operatorname{rot}\vec{v}-\frac{1}{4}\left(\vec{x}^{\perp}\cdot\nabla p-\frac{1}{1-\nu}\left(\vec{x}^{\perp}\cdot(\Delta\vec{v}-\nabla\operatorname{div}\vec{v})-2\operatorname{rot}\vec{v}\right)\right)\right]\\
&= \frac{1}{1-2\nu}\left[\operatorname{rot}\vec{v}-\frac{1}{4}\left(\vec{x}^{\perp}\cdot\nabla p-\frac{1}{1-\nu}\left(\vec{x}^{\perp}\cdot(\nabla p-\nu\nabla p)-2\operatorname{rot}\vec{v}\right)\right)\right]\\
&= \frac{1}{1-2\nu}\left[1-\frac{1}{2(1-\nu)}\right]\operatorname{rot}\vec{v}=\frac{1}{2(1-\nu)}\operatorname{rot}\vec{v}.
\end{aligned}$$

We compose the latter equations with (143) and obtain

$$-\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}+\vec{x}^{\perp}\operatorname{rot}\vec{h}\right)=\frac{1}{4}p\vec{x}-\frac{1}{4(1-\nu)}\vec{x}^{\perp}\operatorname{rot}\vec{v}=\vec{v}-(1-2\nu)\vec{h},$$

which gives (141).

To prove the uniqueness consider

$$-\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}_1+\vec{x}^{\perp}\operatorname{rot}\vec{h}_1\right)+(1-2\nu)\vec{h}_1=-\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}_2+\vec{x}^{\perp}\operatorname{rot}\vec{h}_2\right)+(1-2\nu)\vec{h}_2$$

with harmonic $\vec{h}_1$ and $\vec{h}_2$. Set $\vec{h}_0:=\vec{h}_1-\vec{h}_2$. There follows $\Delta\vec{h}_0=0$ and

$$-\frac{1}{2}\left(\vec{x}\operatorname{div}\vec{h}_0+\vec{x}^{\perp}\operatorname{rot}\vec{h}_0\right)+(1-2\nu)\vec{h}_0=0, \tag{150}$$

from which we calculate by taking divergence and rotation

$$\nu\operatorname{div}\vec{h}_0=0,\ \text{and}\ (1-\nu)\operatorname{rot}\vec{h}_0=0.$$

We have $\operatorname{rot}\vec{h}_0=0$ because $\nu\neq 1$. If $\nu\neq 0$, then we also have $\operatorname{div}\vec{h}_0=0$, which we substitute into (150): $(1-2\nu)\vec{h}_0=0$. Since $\nu\neq\frac{1}{2}$, there also follows $\vec{h}_0=0$. If $\nu=0$, then we have – exactly as in Theorem 1. of [29] – that $\vec{h}_0=\gamma\vec{x}$ for some $\gamma\in\mathbb{R}$. This completes the proof. □

Remark 3.21 Setting $\nu=0$ into Theorem 3.20, we get the representation theorem of two-dimensional Stokes functions from [29]. □

3.3 Connections between the various representations

As pointed out in Remark 2 of [29], the three-dimensional representation formula by Proposition 3.4 for the Stokes functions does not reduce to the two-dimensional formula by Proposition 3.2, although two-dimensional Stokes functions are also Stokes functions in dimension three by setting the third component zero and having the other two components not depending on the third coordinate.

Let the functions $\vec{v} = (v_1(\vec{x}), v_2(\vec{x}))^T \in C_2(\Omega_*)$ and $p = p(\vec{x}) \in C_1(\Omega_*)$ satisfy (129) in some two-dimensional star-shaped domain Ω_*. Then we define by setting $x_3 = 0$ and $v_3(\vec{x}) = 0$:

$$\vec{v}^*(\vec{x}) := (v_1(x_1, x_2, 0), v_2(x_1, x_2, 0), 0)^T \text{ and } p^*(\vec{x}) := p(x_1, x_2, 0).$$

These functions also satisfy (129) in any three-dimensional star-shaped domain Ω, the intersection of which with the plane $x_3 = 0$ is the domain Ω_*. We define $\vec{h}^*$ and ϕ^* by (134), (135) in Theorem 3.17 using the functions $\vec{v}^*$ and p^*. Because we have

$$\vec{x} \cdot \operatorname{rot} \vec{v}^*(\vec{x}) = x_3(\partial_1 v_2(\vec{x}) - \partial_2 v_1(\vec{x})) = 0,$$

for $x_3 = 0$, there follows $\phi^*(x_1, x_2, 0) = 0$. We obtain further

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \times \operatorname{rot} \vec{v}^*(\vec{x}) = \begin{pmatrix} x_2 \operatorname{rot} \vec{v}(\vec{x}) \\ -x_1 \operatorname{rot} \vec{v}(\vec{x}) \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{x}^\perp \operatorname{rot} \vec{v}(\vec{x}) \\ 0 \end{pmatrix}.$$

We substitute this into the definition (134):

$$\vec{h}^*(x_1, x_2, 0) = \frac{2}{3-4\nu} \left\{ \begin{pmatrix} \vec{v}(\vec{x}) \\ 0 \end{pmatrix} - \frac{1}{4} \left[\begin{pmatrix} p(\vec{x})\vec{x} \\ 0 \end{pmatrix} - \frac{1}{1-\nu} \begin{pmatrix} \vec{x}^\perp \operatorname{rot} \vec{v}(\vec{x}) \\ 0 \end{pmatrix} \right] \right\},$$

which we restrict to two dimensions in the obvious way:

$$\vec{h}_*(\vec{x}) = \frac{2}{3-4\nu} \left\{ \vec{v}(\vec{x}) - \frac{1}{4} \left[p(\vec{x})\vec{x} - \frac{1}{1-\nu} \vec{x}^\perp \operatorname{rot} \vec{v}(\vec{x}) \right] \right\},$$

where $\vec{h}_*$ denotes the vector composed from the first and second component of $\vec{h}^*$. If we use Theorem 3.20 with the functions $\vec{v}(\vec{x})$ and $p(\vec{x})$, then we obtain by (143) for the auxiliary harmonic function

$$\vec{h}(\vec{x}) = \frac{1}{1-2\nu} \left\{ \vec{v}(\vec{x}) - \frac{1}{4} \left[p(\vec{x})\vec{x} - \frac{1}{1-\nu} \vec{x}^\perp \operatorname{rot} \vec{v}(\vec{x}) \right] \right\}$$

These formulae show that Theorem 3.17 does not produce the same auxiliary harmonic function as Theorem 3.20 if the functions (129) are restricted to two dimensions. However we have the simple connection

$$\left(\frac{3}{2} - 2\nu\right) \vec{h}_*(\vec{x}) = (1 - 2\nu)\vec{h}(\vec{x}).$$

between these harmonic functions: they are constant multiples of each other.

Remark 3.22 There is another way to observe the connection between the two and three-dimensional representations of the solutions of (129).

Let $\vec{v}$ and p satisfy (129) in a three-dimensional star-shaped domain Ω. Determine the auxiliary harmonic function $\vec{h}(\vec{x}) = (h_1(\vec{x}), h_2(\vec{x}), h_3(\vec{x}))^T$ in Theorem 3.17 by (134) and define

$$\vec{h}^*(\vec{x}) := (h_1(x_1, x_2, 0), h_2(x_1, x_2, 0), 0))^T .$$

Let the corresponding harmonic function (restricted to two dimensions) be $\vec{h}_*(x_1, x_2) := (h_1(x_1, x_2, 0), h_2(x_1, x_2, 0))^T$. We have

$$\operatorname{rot} \vec{h}^*(\vec{x}) = \begin{pmatrix} 0 \\ 0 \\ \operatorname{rot} \vec{h}_*(x_1, x_2) \end{pmatrix} \text{ and } \operatorname{div} \vec{h}^*(\vec{x}) = \operatorname{div} \vec{h}_*(x_1, x_2).$$

Let us substitute $\vec{h}^*$ into the first term of the right-hand side of the three-dimensional formula (132) and set also $x_3 = 0$.

$$\begin{aligned} \vec{x} \operatorname{div} \vec{h}^*(\vec{x}) + \vec{x} \times \operatorname{rot} \vec{h}^*(\vec{x}) &= \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \operatorname{div} \vec{h}_*(x_1, x_2) + \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \operatorname{rot} \vec{h}_*(x_1, x_2) \\ &= \begin{pmatrix} \vec{x}_* \operatorname{div} \vec{h}_*(\vec{x}_*) + \vec{x}_*^{\perp} \operatorname{rot} \vec{h}_*(\vec{x}_*) \\ 0 \end{pmatrix}, \end{aligned}$$

where we also used the notation $\vec{x}_* := (x_1, x_2)^T$. In this way we obtain a two-dimensional restriction of (132), which gives rise to a function

$$\vec{v}_*(\vec{x}_*) := -\frac{1}{2} \left(\vec{x}_* \operatorname{div} \vec{h}_*(\vec{x}_*) + \vec{x}_*^{\perp} \operatorname{rot} \vec{h}_*(\vec{x}_*) \right) + \left(\frac{3}{2} - 2\nu\right) \vec{h}_*(\vec{x}_*),$$

which satisfies along with the corresponding function $p_*(\vec{x}_*) := -2 \operatorname{div} \vec{h}_*(\vec{x}_*)$ the equations

$$\Delta \vec{v}_* = \nabla p_*, \text{ and } \operatorname{div} \vec{v}_* = \left(\nu - \frac{1}{4}\right) p_*$$

in the domain Ω_* which is the intersection of the star-shaped spatial domain Ω with the plane $x_3 = 0$. Hence we obtained that the two-dimensional restriction of the representation formulae for the value $\nu \in \mathbb{R}$ in Theorem 3.17 (in the above described way) gives rise to formulae corresponding to Theorem 3.20 for the value $\nu - \frac{1}{4}$ instead of ν. (Observe also that $\nu \neq \frac{3}{4}$ in Theorem 3.17 while $\nu \neq \frac{1}{2}$ in Theorem 3.20.) $\square$

A Summary

This thesis concerns the first kind Stokes problem in fluid mechanics with a stress on the study of the properties of the so-called inf-sup constant, which assures the stable solvability of the Stokes problem, and which depends only on the shape of the domain on which the problem is to be solved. The related Schur complement operator of the first kind Stokes problem is also studied. The thesis is built up of two sections each of them containing several subsections.

In the introduction we show how the Stokes problem is derived from the Navier-Stokes equations, which are the fundamental equations of motion of a viscous fluid. We explain the scope and the structure of the thesis.

In the first section we investigate the first kind Stokes problem posed on two-dimensional simply connected domains and hence we exploit the usage of conformal mapping. We reprove the connection between the Schur complement operator and the Friedrichs operator of the domain. The domain dependence of these operators is also explained in a class of domains having sufficiently smooth boundary. Next, we examine the spectra of these operators. An element of the spectrum of the Schur complement operator is the inf-sup constant of the domain which is particularly examined. We give results on the multiplicity of the eigenvalues of the studied operators depending on symmetric geometric properties of the domain. The domain dependence of the inf-sup constant is explained, and estimations of its value are given in terms of the related conformal mapping. Numerous examples are given, which illustrate the theory for specific domains (and for specific conformal mappings). Some examples are, in addition, of theoretical importance concerning the continuous domain dependence of the inf-sup constant and also its multiplicity. We give some results and examples also for domains the boundary of which has corners. In this case a continuous part appears in the corresponding spectra. At the end of the first section we touch the case of multiply connected domains. We are able to establish the same connection between the studied operators as for simply connected domains provided the boundary curves of the multiply connected domain are smooth enough.

In the second section we investigate several representation formulae for Stokes functions in two- and also in three-dimensional domains. First, the connection of these formulae to the results of the previous section is briefly explained in the case of two-dimensional domains. Next, the connections between the already known representations are examined. We generalize the representation results for the equations of linear elasticity. Finally, the connections between the two- and three-dimensional representation formulae are explained.

B Összegzés

Ezen disszertáció az elsőfajú Stokes feladatot vizsgálja különös tekintettel az ún. inf-sup konstansra, amely fontos szerepet játszik a Stokes feladat stabil megoldhatósága szempontjából és számértéke szoros összefüggésben van azon tartomány alakjával, amelyben a feladatot kitűztük. Ugyancsak vizsgáljuk az elsőfajú Stokes feladathoz tartozó Schur komplement operátort. A dolgozat két fő részből épül fel, melyek további alfejezetkre tagolódnak.

A bevezetésben röviden felidézzük, hogy, kiindulva a Navier-Stokes egyenletrendszerből, hogyan juthatunk el a Stokes feladathoz. Megfogalmazzuk a dolgozat célkitűzéseit és bemutatjuk a felépítését.

Az első részben egy síkbeli tartományon kitűzött elsőfajú Stokes feladatot vizsgáljuk felhasználva a konform leképezések elméletéből adódó lehetőségeket. Ezen eszközökkel kapcsolatot mutatunk ki a Schur komplement operátor ill. a tartományhoz rendelhető Friedrichs operátor között, amennyiben a tartomány pereme megfelelően síma. Eredményt fogalmazunk meg ezen két operátornak a tartomány alakjától való folytonos függésére is. Tanulmányozzuk az operátorok sajátérték feladatait (az inf-sup konstans négyzete a legkisebb pozitív sajátértéke a Schur komplement operátornak). Kimutatjuk, hogy a sajátértékek multiplicitása kapcsolatban van a síkbeli tartomány szimmetriáival . Feltételeket fogalmazunk meg az inf-sup konstans tartomány alakjától való folytonos függésére, valamint becslést adunk az értékére. Számos példán szemléltetjük az elméleti eredményeket. Megfogalmazunk néhány eredményt olyan tartományok esetére is, melyek pereme egyenként síma, egymáshoz adott szögben kapcsolódó görbékből áll. Ebben az esetben egy folytonos rész is megjelenik a kapcsolódó operátorok spektrumában. Az első rész végén külön foglalkozunk a többszörösen összefüggő tartományok esetével. Kimutatjuk a tanulmányozott operátorok kapcsolatát (mint az egyszeresen összefüggő esetben), feltéve, hogy a tartomány peremgörbéi megfelelően símák.

A második részben a Stokes függvények reprezetációjával foglalkozunk síkbeli és térbeli tartományok esetében egyaránt. Először kapcsolatba hozzuk az ezen részbeli reprezentációs képleteket az előző résszel, amennyiben a vizsgált tartomány síkbeli. Majd a szakirodalomban már ismert reprezentációs eredmények között fennálló kapcsolatokat mutatjuk ki. Ezen reprezentációs eredményeket általánosítjuk a lineáris rugalmasságtani egyenlet megoldásaira. A második rész végén, felhasználva az általánosított képleteket, kapcsolatot mutatunk ki a síkbeli illetve térbeli reprezentációs eredmények között.

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