Parity problems of combinatorial polymatroids

Márton Makai

Ph.D. dissertation

Eötvös University, Faculty of Science

Doctoral school: Mathematics
Director: Professor Miklós Laczkovich
member of the Hungarian Academy of Sciences

Doctoral program: Applied mathematics
Director: Professor András Prékopa
member of the Hungarian Academy of Sciences

Supervisor: Zoltán Király, Ph.D., associate professor

The dissertation was written at the
MTA-ELTE Egerváry Research Group on Combinatorial Optimization

February 2009

Contents

Acknowledgments

I am greatly indebted to my supervisor Zoltán Király for always inspiring my doctoral work. Besides, he helped me to work in industrial projects e.g. in the Communication Networks Laboratory which gave me also an indispensable financial support for my studies.

I am grateful to András Frank, Jácint Szabó, Gyula Pap, and Tamás Király with whom I worked on many of my results. They and the other members of the Egerváry Research Group gave a highly excellent work atmosphere. It is a rare joy to work in such an environment.

I would like to thank András Sebő for his altruistic help during the half a year that I spent in Grenoble with an ADONET doctoral fellowship.

Finally, it is difficult to put in words, how thankful am I to my family, especially to Zsuzsi for giving me always the support I needed, I would like to thank for their patience.

Chapter 1

Preliminaries

1.1 Background and motivation

The matching problem of graphs with the fundamental works of Tutte, Berge, Gallai, and Edmonds plays a central role in the history of combinatorial optimization. Another fundamental field of combinatorial optimization starting from flows, connectivity, matroids, and polymatroids led to the development of another wide theory related to submodularity. The matching problem of graphs and the matroid intersection problem (which are both essential problems from these fields) suggested Lawler to introduce a common generalization: the matroid matching problem (a.k.a. polymatroid matching, matroid parity, or polymatroid parity) [30; 31]. Equivalent definitions were given by Edmonds and Jenkyns [25].

For a long time all the solvable special cases of matroid parity turned out to be reducible to the above mentioned two special cases. Thereafter came Lovász' seminal results on matchings of polymatroids [33] including his characterization for the size of the maximum matching for linear polymatroids [34]. He developed also an algorithm for the linearly represented case [35; 39]. The value of this result is in the common experience that most of the matroids that we meet in daily life are linear. However, the linear approach is not satisfying for combinatorial problems as the techniques and the solution obtained does not reflect the combinatorial nature of the problem. The phenomenon described by the last sentence is one of the most important motivations of this work where we try to have a more combinatorial viewpoint.

One remark is needed right here on the notion of “being combinatorial”, which may be also a partial explanation of the title. One might say that matroids and polymatroids are purely combinatorial structures, so what do we mean by drawing the distinction between combinatorial and non-combinatorial polymatroids? Our applications usually come from graph theory, and from the combinatorial optimization inseminated by Edmonds’ notion of good characterization. Hence, in our point of view, polymatroids defined by natural finite combinatorial objects i.e. by graphs, hypergraphs, cuts, trees, by their substructures, etc. are considered to be combinatorial. We do not think a full linear matroid, or a polymatroid defined by the transcendence degrees of subfields of a field extension as a really combinatorial object. Though the distinction is not very exact, we hope that our view will be clear for the reader.

But let us go back to our main line of outlining what happened in the parity history after Lovász’ results. Just after that Lovász [33] solved the linear case, he and independently Jensen and Korte [26] showed that the matching problem is not tractable for general matroids. Thus, the main problem which remained is to explore wider classes of matroids where the size of the maximum matching has a good characterization. In order to characterize the size of the maximum matching, Lovász’ structure theorem either decomposes the problem into smaller ones or shows the existence of some special substructures. In the latter case we can get stuck. One of these substructures are the so called non-trivial double circuits. According to the present state of the theory, the nice behavior of the double circuits is responsible for the existence of a good characterization. This observation was made explicit by Dress and Lovász [10]. They introduced the double circuit property (DCP) as a possible way of this good behavior, which seemed to be an important property of linear matroids in Lovász’ proof. We got therefore that the size of the maximum matching of DCP polymatroids have a good characterization. As the place of DCP matroids in the hierarchy of matroids is not evident, Björner and Lovász [3] and Hochstättler and Kern [23] examined abstract properties like series reduction property or pseudomodularity which imply the DCP. Unfortunately, the study of matroid classes related to the DCP stopped in the late eighties, and the obtained results helps only in few cases of our applications. Therefore, it remains a great challenge to explore combinatorially suggested polymatroid classes where we have a good characterization.

There are various problems in combinatorial optimization which can be formulated naturally as matroid matching. Lovász [33] has shown also several examples where this nature of the problem is less apparent. These are the maximum triangle cactus problem of 3-uniform hypergraphs, Mader's vertex-disjoint $\mathcal{A}$-paths problem, and the pinning down problem of 2-dimensional bar-and-joint frameworks. The first algorithmic approach to the maximum genus embedding problem of graphs is based on the matroid parity problem of the cographic matroid [16]. The latter problem can be generalized in several ways, one of these leads to the theory of parity constrained connectivity orientations of graphs [13; 15; 27; 44]. Matroid matching appears even in the field of approximation algorithms, the best approximation for the planar subgraph problem [6; 7; 8] is based on maximum triangle cacti, and there are valuable implications in the Steiner tree approximations [2; 47; 48].

If most of the everyday polymatroids (including the polymatroids arising in the above mentioned applications) are linear, what are the problems with that approach? We will see polymatroids in applications, such that we would not bet that all of them are linear. We have already mentioned that even if the polymatroid is linear, the min-max relations and characterizations obtained by linearity do not have a combinatorial nature. Another problem which may happen is that we are not aware of a deterministic method computing a representation. In this case we do not get a good characterization of the size of the maximum matching nor an algorithm computing it unless randomization is allowed. We think that the motivation of Dress and Lovász to introduce the DCP was to extend Lovász min-max relation to wider classes than linear polymatroids. For us, their result is more interesting for some subclasses of linear polymatroids. By the help of the DCP, more combinatorial characterizations will be obtained for some combinatorial polymatroids. We will see also characterizations which are not likely to be related to polymatroids having the DCP.

Although it turns out that the topic of this thesis is related to all of the typical topics of combinatorial optimization ranging from good characterizations, deterministic and randomized algorithms, heuristics to approximation algorithms, our work is restricted to the first few members of this list. By being solvable we usually mean a simple good characterization to the problem, we are trying to eliminate randomization, and in very few cases we also mention how to work out a polynomial deterministic combinatorial algorithm by tweaking the known algorithms.

We have to mention that the most recent results related to parity deal with the parity problem of delta matroids (or more generally of jump systems) [4; 5; 19; 20; 37]. As very little is known about the combinatorial instances of these problems where combinatorial characterizations exist, the present work absolutely omits this topic.

1.2 Overview of the thesis

The thesis discusses the matching problem of polymatroids arising in combinatorial applications, and introduces polymatroid constructions which hopefully give a more unified view of these problems.

Most of the results in the theoretical part of matroid parity literature heavily rely on Lovász' papers, lacking therefore being self-contained. We try to summarize the most important preliminary results and their proofs. To account for this we emphasize that these statements can be stated – and in fact are stated in the literature – in many different forms. It would be messy even to cite the results and retransform them to the appropriate forms. We devote Section 2 to work out this, which is hopefully useful for those readers who are not familiar with the related literature.

In the sequel we start a detailed examination of the parity of some combinatorial matroids and polymatroids. It is an important common feature of our polymatroids that they are usually defined by intersecting/crossing sub-, or supermodular functions. Although each polymatroid (or base polyhedron) can be defined this way, this may endow the polymatroid with interesting properties. The evident question is that under what properties of the defining intersecting submodular function does the arising matroid have the DCP, or some other properties which imply the existence of a good characterization for the size of the maximum matching.

Our starting point is the examination of count matroids [57; 58] in Chapter 3. In the thesis they are referred as (k, l)-matroids, according to their two parameters. For some special cases we already know the DCP. This is the case for cycle matroids of complete graphs ($k = l = 1$), and for transversal matroids ($k = 1$, $l = 0$) if each singleton is in the ground set. These imply the Berge-Tutte formula [1], the min-max relation for transversal matroid matching [10; 55], graphic matroid matching (maximum triangle cactus) [33]. We prove that a density condition for the hypergraph (which forms the ground set of the matroid) imply the DCP for general (k, l)-matroids. It is an interesting corollary that the 2-dimensional generic rigidity matroid of the complete graph has the DCP. There is also a consequence concerning an estimation of the rank of the 3-dimensional generic rigidity matroid.

In Chapter 4 we generalize this by introducing a new concept, the class of solid polymatroids. These polymatroids are defined again by intersecting submodular functions, and we extract some properties of count matroids which were needed to the DCP. The construction involves the min-max relation for the maximum number of Mader's vertex-disjoint $\mathcal{A}$-paths [41] (Chapter 5), and the characterization for the parity constrained orientation problem of Frank, Jordán, and Szigeti [13] (Subsection 6.2.1). There will be other important implications in the field of parity constrained connectivity orientations which is the topic of the subsequent chapter.

A rather new area of parity problems arises in parity constrained connectivity orientation problems of graphs and hypergraphs [13; 15; 27; 44] (Chapter 6). It is known that if a connectivity requirement is described by an intersecting or crossing supermodular function, then the class of graphs or hypergraphs having an orientation covering the requirement can be characterized. We ask for the existence of orientations having moreover a prescribed parity of out-degree for each vertex. Hence, this is a possible generalization of matchings of graphs and connectivity orientations. We will see that some special cases of this problem fit into the framework of solid polymatroids, but Király and Szabó's [27] general problem where the connectivity requirement is described by a non-negative intersecting supermodular function does not seem to. To insert this problem into the theory of polymatroid parity, a new phenomenon must be exploited (Chapter 7). First, it is an important observation that these polymatroids have no non-trivial compatible double circuits (NTCDCs) at all. Second, it will be shown also that for polymatroids without NTCDCs a partition type formula characterizes the maximum matching. The class of polymatroids without NTCDCs is neither a sub- nor superclass of DCP polymatroids, the members are unlikely to be linear or to have structural properties related to the DCP. This class shows that matroid parity is not a closed, finished theory; the search for connections between DCP polymatroids and polymatroids without NTCDCs may open new research topics of matching theory.

Very little is known about the matching problem of polymatroids constructed by crossing submodular functions in terms of the properties of the defining crossing submodular function. We present a characterization for planar graphs having strongly connected orientations with even out-degrees (Subsection 6.3.1). But we have no result for the problem formulated to general graphs.

Most of the algorithmic questions related to the characterizations are left open by this study. We mention some cases when we are aware of an algorithm computing the maximum matching and the combinatorial dual proof, but most of them are left for future challenges (see Chapter 8).

1.3 Submodular functions, matroids, and polymatroids

For the most important graph, hypergraph, and matroid theoretical concepts and notions the reader is referred to standard textbooks, e.g. to [50; 51; 52]. Here we present the most important properties of polymatroids as the basic results about matchings are presented for polymatroids. Matroids are tacitly considered as special cases of polymatroids and we follow this principal even when some notations happen to be unusual.

Let S be a ground-set. A set-function $b : 2^S \to \mathbb{Z} \cup \{\infty\}$ is said to be *submodular* if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{1.1}$$

holds for every $X, Y \subseteq S$. Similarly, $p : 2^S \to \mathbb{Z} \cup \{-\infty\}$ is said to be *supermodular* if $-p$ is submodular, i.e., if p satisfies (1.1) with the opposite inequality sign. $m : 2^S \to \mathbb{Z}$ is said to be *modular* if m is both submodular and supermodular, i.e., m satisfies (1.1) with equality.

A set-function b is *non-decreasing* if $b(X) \leq b(Y)$ whenever $X \subseteq Y \subseteq S$, and *non-increasing* if $-b$ is non-decreasing. The set-function b is said to be *finitely generated* if for every $X \subseteq S$ with finite $b(X)$, there exists a finite $Y \subseteq X$ with $b(Y) \geq b(X)$. A non-decreasing finitely generated upper-bounded submodular set-function $f : 2^S \to \mathbb{Z}$ with $f(\emptyset) = 0$ is said to be a *polymatroid function*.

Normally, polymatroids and matroids are defined with a finite ground-set. In some cases however we have to deal with infinite matroids or polymatroids, the most important among those is the *full linear matroid*. For sake of simplicity, the ground-set is supposed to be finite unless the polymatroid is a linear one. If $f(s) \leq k$ for every $s \in S$ and some integer k, then we speak about a *k-polymatroid function*.

If $x : S \to \mathbb{R}$ and $U \subseteq S$, then let us use the notation $x(U) = \sum_{s \in U} x(s)$. Hence, x naturally extends to a $2^S \to \mathbb{R}$ function. If S is finite, then the polyhedra

$$\mathcal{P}(f) = \{x \in \mathbb{R}_+^S : x(U) \leq f(U) \text{ for every } U \subseteq S\}$$

and

$$\mathcal{B}(f) = \{x \in \mathbb{R}_+^S : x(U) \leq f(U) \text{ for every } U \subseteq S,\ x(S) = f(S)\}$$

associated with the polymatroid function f are called the *polymatroid* of f and the *base polyhedron* of f resp. For infinite S, these definitions are not totally correct, however, in any case when we deal with infinite polymatroids, the vectors will be restricted to have finite support.

If $x : S \to \mathbb{Z}_+$, then we can be interested in the vector $y : S \to \mathbb{Z}_+$, $y \leq x$ with $y \in \mathcal{P}(f)$ which maximizes $y(S)$. It is well-known that

$$\max\{y(S) : y \leq x,\ y \in \mathcal{P}(f)\} = \min_{U \subseteq S}(x(S - U) + f(U)). \tag{1.2}$$

This quantity is called the *rank* of x, and it is denoted by $r_f(x)$. We also have to note that the rank function determines the polymatroid. For $x \in \mathbb{Z}_+$ we let $\mathrm{def}_f(x) = x(S) - r_f(x)$, the *defect* or *deficiency* of x. The rank has the submodular property

$$r_f(x) + r_f(y) \geq r_f(x \wedge y) + r_f(x \vee y),$$

where $\vee$ and $\wedge$ stands for the coordinate-wise maximum and minimum resp.

The (unique) maximal set $U \subseteq S$ which gives equality in (1.2) is called the *span* of x, and it is denoted by $\mathrm{sp}_f(x)$. It is also known that $\mathrm{sp}_f(x) = \{s \in S : r_f(x + \chi_s) = r_f(x)\}$. The subsets of S arising as $\mathrm{sp}_f(x)$ for some vector x are called the *flats* of f. Then, $U \subseteq S$ is a flat, if and only if $f(U \cup \{s\}) > f(U)$ for each $s \in S - U$.

A 1-polymatroid is simply called *matroid*, and a 1-polymatroid function is also called as a *matroid rank function*. If $f : 2^E \to \mathbb{Z}_+$ is a matroid rank function, then $f(F) = r_f(\chi_F)$ for every $F \subseteq E$.

1.4 Polymatroid operations and constructions

Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function. First we recall some polymatroid operations, which in fact can be derived from the analogous operations of matroids. Therefore, let us start our list with the homomorphic map operation a.k.a. homomorphic image which gives a way to construct every polymatroid from a matroid.

(1.3i) **Homomorphic image and prematroids of polymatroids.** Let $\psi : S \to R$ be a function. Now $\psi(f) : 2^R \to \mathbb{Z}_+$, $U \mapsto f(\psi^{-1}(U))$ is the *homomorphic image* of f under ψ. Coherently to the above notations, if $n : S \to \mathbb{Z}$, then by $\psi(n)$ we mean the vector in $\mathbb{Z}^R$ having $\psi(n)(r) = \sum_{s\in\psi^{-1}(r)} n(s)$.

It is clear that matroid rank functions on E are exactly the polymatroid functions $f : 2^E \to \mathbb{Z}_+$ having $f(F) \le |F|$ for every $F \subseteq E$. But there is an even closer relation between polymatroids and matroids. For the polymatroid function f, it is possible to define a k-polymatroid function g, the homomorphic image of which is f, s.t. g is the "most independent" in some sense [22]. The ground set E of g is the disjoint union of sets E_s for $s \in S$ with sizes $|E_s| \ge f(\{s\})/k$. Let $\varphi : E \to S$ with $\varphi(e) = s$ if $e \in E_s$, and

$$g(F) = \min_{Y\subseteq F}(k|F - Y| + f(\varphi(Y)) = \min_{U\subseteq S}(k|F - \bigcup_{s\in U} E_s| + f(U)).$$

It is a routine to prove that g is a k-polymatroid function, $\varphi(g) = f$, and for $x \in \mathbb{Z}_+^E$, $x \le k$ we have $x \in \mathcal{P}(g)$ if and only if $\varphi(x) \in \mathcal{P}(f)$. The polymatroid obtained is called a *pre-k-polymatroid* of f. For $k = 1$ we speak about *prematroids*. Note that a pre-k-polymatroid is uniquely determined by f, k, and by the sizes $|E_s|$, $s \in S$; and any of the prematroids determines the original polymatroid. If $\mathcal{M}$ is any matroid with rank function r then the prematroids of r are the parallel extensions of $\mathcal{M}$. If we consider a prematroid $\mathcal{M}$ then we tacitly assume that $\mathcal{M}$ and the function $\varphi : E \to S$ is given.

After applying a matroid operation ζ to a prematroid $\mathcal{M}$ of a polymatroid, we can consider the polymatroid, the prematroid of which is $\zeta(\mathcal{M})$.

Therefore, ζ induces a polymatroid operation for most of the usual matroid operations.

(1.3ii) **Restriction or deletion.** If $U \subseteq S$, then $f|_{2^U}$ is a polymatroid function again. The new polymatroid function $f|_{2^U}$ is said to be the *restriction* of f to U, we say that it arises from f by *deleting* $S - U$.

(1.3iii) **Translation.** If $n \in \mathbb{Z}_+^S$ then $f + n : 2^S \to \mathbb{Z}_+$, $U \mapsto f(U) + n(U)$ is a polymatroid function. It is clear that $\mathcal{B}(f + n) = \mathcal{B}(f) + n$.

(1.3iv) **Deletion or upper bound.** Let $u \in \mathbb{Z}_+^S$ be a bound vector. Then, it can be checked that the polyhedron $\mathcal{P}(f) \cap (u + \mathbb{R}_-^S)$ is a polymatroid, and it is determined by the polymatroid function $f \backslash u = \varphi(r_{\mathcal{M}|Z})$ where $\mathcal{M}$ is a prematroid of f and $Z \subseteq E$, $\varphi(\chi_Z) = u$. The matroid union theorem asserts that $f \backslash u$ is indeed a polymatroid function and $(f \backslash u)(U) = \min_{Y \subseteq U}(f(Y) + u(U - Y))$. If $\mathcal{M}$ is a matroid and $u \in \{0,1\}^S$ then $(r_{\mathcal{M}} \backslash u)|_{2^{\mathrm{supp}(u)}} = \mathcal{M}|\mathrm{supp}(u)$.

(1.3v) **Direct sum.** The direct sum of the polymatroid functions $f_i : 2^{S_i} \to \mathbb{Z}_+$, $i \in \{1, 2\}$, where S_1, S_2 are disjoint sets, is $f_1 \oplus f_2 : 2^{S_1 \dot\cup S_2} \to \mathbb{Z}_+$, $U \mapsto f_1(U \cap S_1) + f_2(U \cap S_2)$.

(1.3vi) **Dual.** If $u \in \mathbb{Z}_+^S$ and $u_s \geq f(\{s\})$ for every $s \in S$, then f_u^* is defined through $\mathcal{B}(f_u^*)$ which is the reflection of $\mathcal{B}(f)$ to $u/2$. Clearly $f_u^*(U) = u(U) - f(S) + f(S - U)$ for $U \subseteq S$. If $\mathcal{M}$ is a matroid with ground-set S and rank function r, then $r_{\mathbf{1}}^*$ is the rank function of $\mathcal{M}^*$, where $\mathbf{1}$ denotes the everywhere 1 vector.

(1.3vii) **Sum.** The sum of $f_i : 2^S \to \mathbb{Z}_+$, $i \in \{1, 2\}$ is simply $f_1 + f_2$. Now it can be shown that $\mathcal{B}(f_1 + f_2) = \mathcal{B}(f_1) + \mathcal{B}(f_2)$. If $\mathcal{M}_i$ are matroids with rank function r_i, $i \in \{1, 2\}$, then $\mathcal{M}_1 + \mathcal{M}_2$ has rank function $(r_1 + r_2) \backslash 1$. The sum is a special case of the homomorphic image: if we think that the ground sets S_i $(= S)$ of f_i are disjoint then $f_1 + f_2 = \psi(f_1 \oplus f_2)$ where $\psi(s_i) = s$ for $s \in S$.

(1.3viii) **Contraction or lower bound.** Contraction is defined through the prematroid. Let $z \in \mathbb{Z}_+^S$, and let $\mathcal{M}$ be a prematroid with $|E_s| \geq z(s)$, and

$Z \subseteq E$ with $\varphi(\chi_Z) = u$. Then, $f/z = \varphi(\mathcal{M}/Z)$. If $l \in \mathbb{Z}_+^S \cap \mathcal{P}(f)$ is a vector below z with the largest sum of coordinates, then $\mathcal{B}(f/z) = (\mathcal{B}(f) - l) \cap \mathbb{R}_+^S$. It is well-known that f/z is a polymatroid function and

$$(f/z)(U) = \min_{Y \supseteq U}(f(Y) - l(Y)) = f(U \cup \mathrm{sp}_f(z)) - f(\mathrm{sp}_f(z)) - z(U - \mathrm{sp}_f(z)).$$

We say that the arising polymatroid function is obtained by *contracting* z. It is also clear, that $r_{f/z}(x) = r_f(x+z) - r_f(z)$.

If $\mathcal{M}$ is a matroid and $z \in \{0,1\}^S$ then $r/z = r_{\mathcal{M}/Z}$ where $\varphi(\chi_Z) = z$.

Let $Z \subseteq E$. Then, the polymatroid function $U \mapsto f(U \cup Z) - f(Z)$ arises as a special case of the contraction operation by taking $z = f(Z)\chi_z$.

(1.3ix) **Projection.** Loosely speaking, projections are contractions where the contracted element is not a member of the ground set. But more precisely, let $f', f : 2^S \to \mathbb{Z}_+$ be polymatroid functions. Then we say that f' is a *projection* of f if $f - f'$ is non-decreasing. If $f(X) - f'(X) \leq 1$ for every $X \subseteq S$, then we say that the projection is a 1-*projection.* For example, contractions are projections. More specially, for $f(s) > 0$ and $f' = f/\chi_s$, i.e.

$$f'(X) = \begin{cases} f(X) - 1, & \text{if } f(X \cup \{s\}) = f(X), \\ f(X), & \text{otherwise,} \end{cases}$$

f' is a 1-projection of f. Every projection arises by consecutive application of 1-projections. Let $f_i(X) = \min(f'(X) + i, f(X))$, where $i = 0, 1, \ldots, k = f(S) - f'(S)$. Then, $f_k = f$, $f_0 = f'$, the f_i's are polymatroid functions, and f_i is a 1-projection of f_{i+1}. In fact, each projection arises as a restriction of a contraction of a polymatroid extending the original one.

Claim 1.4.1. *If $f, f' : 2^S \to \mathbb{Z}_+$ are polymatroid functions, and f' is a projection of f, then there exists a polymatroid function $\tilde{f} : 2^{\tilde{S}} \to \mathbb{Z}_+$, $\tilde{f}|_{2^S} = f$ s.t. $f' = (\tilde{f}/z)|_{2^S}$ for some $z : \tilde{S} \to \mathbb{Z}_+$.*

Proof. By the above note, we have to deal only with 1-projections. Then, let $\tilde{S} - S = \{s\}$, and

$$\tilde{f}(X) = \begin{cases} f(X), & \text{if } s \notin X, \\ 1 + f'(X - \{s\}), & \text{otherwise.} \end{cases}$$

□

(1.3x) **Truncation.** Given a bound k, let $f^k(U) = \min(f(U), k)$. f^k is equal to $\varphi(r(\mathcal{M}^k))$ where $\mathcal{M}^k$ is the k-truncation of the prematroid $\mathcal{M}$ of f.

(1.3xi) **Dilworth truncation.** Let S be finite, and let $f : 2^S \to \mathbb{Z}_+$ be positive on non-empty sets. Then, $\widehat{f}(U) = \sum_{U=U_1\dot\cup U_2\dot\cup\ldots\dot\cup U_t,\ U_i\neq\emptyset}(f(U_i) - 1)$, if $U \neq \emptyset$. Then, $\widehat{f}$ is a polymatroid function, called the *Dilworth truncation* of f.

(1.3xii) **Principal extension.** This operation is to add a new element to the ground-set of the polymatroid which is in general position in a flat. Hence, let $U \subseteq S$ be a flat, $0 < k \leq f(U)$, and we associate a new member u to U. Then, let $f' : 2^{S\cup\{u\}} \to \mathbb{Z}_+$, s.t. $f'|_{2^S} = f$; and for $X \subseteq S$ let $f'(X \cup \{u\}) = f(X) + k$ if $f(X \cup U) > f(X)$, and $f'(X \cup \{u\}) = f(X)$ if $f(X \cup U) = f(X)$.

(1.3xiii) **Polymatroids from intersecting submodular functions.** The construction of polymatroids with Dilworth truncation suggests a more general one, which is again well-known from the literature [9; 11; 51]. Let S be finite, let $\emptyset \in \mathcal{L} \subseteq 2^S$ be a family which is closed under taking intersections, and $\bigcup \mathcal{L} = S$. Let $b : \mathcal{L} \to \mathbb{Z}_+$, $b(\emptyset) = 0$ be a function having the following *intersecting submodular* property. If $U_1, U_2 \in \mathcal{L}$ have non-empty intersection, then let us assume the existence of a member of $\mathcal{L}$ denoted by $U_1 \vee U_2$ s.t. $U_1 \cup U_2 \subseteq U_1 \vee U_2$, and

$$b(U_1) + b(U_2) \geq b(U_1 \cap U_2) + b(U_1 \vee U_2).$$

Then, $\widehat{b} : 2^S \to \mathbb{Z}_+$,

$$\widehat{b}(U) = \min_{\mathcal{F}\subseteq\mathcal{L}-\{\emptyset\},\ U\subseteq\bigcup\mathcal{F}} \sum_{U_i\in\mathcal{F}} b(U_i) \tag{1.4}$$

is a polymatroid function.

Choosing $\mathcal{L} = 2^S$, and $b(U) = f(U) - 1$ whenever $U \in \mathcal{L} - \{\emptyset\}$, we get back (1.3xi).

For $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{L} - \{\emptyset\}$, let us say that $\mathcal{F}_1$ is a *refinement* of $\mathcal{F}_2$, if for each $U_1 \in \mathcal{F}_1$, there exists $U_2 \in \mathcal{F}_2$ with $U_1 \subseteq U_2$. If $U \subseteq S$, then there exists a unique family $\mathcal{F}_U$ which gives equality in (1.4), and each $\mathcal{F}$ giving equality in (1.4) refines $\mathcal{F}_U$. It is clear that $\mathcal{F}_U$ is composed by pairwise disjoint sets. If $U_1 \subseteq U_2 \subseteq S$, then $\mathcal{F}_{U_1}$ refines $\mathcal{F}_{U_2}$. For want of better name, the latter construction is also called Dilworth truncation.

As the polymatroid functions obtained by this construction have a particular importance in our study, it is worthwhile to say more about their contractions. In fact, the contractions are again of this type. Let $z \in \mathbb{Z}_+^S$. Let

$$\mathcal{L}_z = \{U \in \mathcal{L} : U \cap Z \in \{\emptyset, Z\} \text{ for every } Z \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}\},$$

and let us define $b_z : \mathcal{L}_z \to \mathbb{Z}_+$ by

$$b_z(U) = b(U) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U]} b(X) - z(U - \mathrm{sp}_{\widehat{b}}(z)).$$

We only have to define the operation $\vee_z$. Thereafter, it will be clear that $\widehat{b_z} = \widehat{b}/z$.

Let $U_1, U_2 \in \mathcal{L}_z$ with $U_1 \cap U_2 \neq \emptyset$. By applying submodularity to b, we get

$$\begin{aligned}
&b_z(U_1) + b_z(U_2) \geq \\
&\qquad b(U_1 \cap U_2) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_1 \cap U_2]} b(X) - z(U_1 \cap U_2 - \mathrm{sp}_{\widehat{b}}(z)) + \\
&\qquad b(U_1 \vee U_2) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_1 \cup U_2]} b(X) - z(U_1 \cup U_2 - \mathrm{sp}_{\widehat{b}}(z)) \geq \\
&b_z(U_1 \cap U_2) + b(U_1 \vee U_2) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_1 \vee U_2]} b(X) - z(U_1 \vee U_2 - \mathrm{sp}_{\widehat{b}}(z)).
\end{aligned}$$

It is not hard to see (using submodularity) that $U_1 \vee U_2$ can be replaced by a member of $\mathcal{L}_z$:

Proposition 1.4.2. *If $U \in \mathcal{L}$, then there exists $U' \in \mathcal{L}_z$ s.t. $U \subseteq U'$ and*

$$b(U) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U]} b(X) - z(U - \mathrm{sp}_{\widehat{b}}(z)) \geq b_z(U').$$

Hence,

$$b_z(U_1) + b_z(U_2) \geq b_z(U_1 \cap U_2) + b_z((U_1 \vee U_2)')$$

for the set $(U_1 \vee U_2)'$ given by Proposition 1.4.2.

For $Z \subseteq S$, by $\mathcal{L}_Z$ and b_Z, we mean $\mathcal{L}_z$ and b_z with $z = \widehat{b}(Z)\chi_Z$.

Chapter 2

Introduction to polymatroid matching

This chapter is devoted to present the most important notions and statements from the basics of polymatroid parity. Most of the results are from [10; 33; 34; 35] even if they are presented in a technically different way.

2.1 Matchings and covers of polymatroids

In what follows, S is a finite ground-set and $f : 2^S \to \mathbb{Z}_+$ is a polymatroid function. We use the shortened notations $r = r_f$, $\mathrm{sp} = \mathrm{sp}_f$, whenever this does not cause ambiguities. An even vector $m : S \to \mathbb{Z}_+$ is said to be a *polymatroid matching* or shortly *matching* if $m \in \mathcal{P}(f)$. Let

$$\nu(f) = \max\{m(S)/2 : m \text{ is a matching of } f\}.$$

In parallel with this notion, the even vector c is said to be a *polymatroid cover* or shortly *cover*, if $r(c) = f(S)$. Let

$$\varrho(f) = \min\{c(S)/2 : c \text{ is a cover of } f\}.$$

Before going further, as an illustration let $G = (V, E)$ be an undirected graph without isolated vertices, and let $q : 2^E \to \mathbb{Z}$ be the set-function s.t. $q(F) = |\bigcup F|$ for every $F \subseteq E$, i.e., the number of end-vertices of the edges of F. It can be seen easily that q is a 2-polymatroid function. Moreover, if $M \subseteq E$ is a matching of the graph G, then $2\chi_M$ is a polymatroid matching of q. Similarly, the supports of polymatroid matchings of q are exactly the matchings of the graph. Furthermore, $\varrho(q)$ gives the minimum number of edges covering all the vertices, i.e., an edge cover in standard graph theoretic terminology. The relation of $\nu(q)$ and $\varrho(q)$ is well-known by Gallai's classical theorem [18]:

Theorem 2.1.1 (Gallai)**.** *For any undirected graph $G = (V, E)$ without isolated vertices, we have*

$$\nu(q) + \varrho(q) = |V|.$$

The two quantities are related similarly for general polymatroids:

Theorem 2.1.2 (Lovász)**.** *For any polymatroid function $f : 2^S \to \mathbb{Z}_+$, we have*

$$\nu(f) + \varrho(f) = f(S).$$

Claim 2.1.3. *For any cover c of f, there exists a matching $m \leq c$ with $m(S)/2 \geq f(S) - c(S)/2$.*

Proof. Let $m = c$ at the beginning. Let us reset m to $m - 2\chi_s$ as long as there exists $s \in \operatorname{supp}(m)$ with $r(m - 2\chi_s) \geq r(m) - 1$. Finally, m will be a matching. Indeed, let m' be a maximum matching having $m' \leq m$ and let $s \in S$ with $m(s) - m'(s) > 0$. Then, $r(m - 2\chi_s) \leq r(m) - 2$, and this implies $r(m - k\chi_s) \leq r(m) - k$, where $k = m(s) - m'(s)$. By submodularity,

$$r(m' + k\chi_s) + r(m - k\chi_s) \geq r((m' + k\chi_s) \wedge (m - k\chi_s)) + r((m' + k\chi_s) \vee (m - k\chi_s)) = r(m') + r(m) \geq r(m') + r(m - k\chi_s) + k,$$

contradicting the maximality of m'. Hence, m is a matching, $r(m) \geq f(S) - (c(S)/2 - m(S)/2)$, and $m(S)/2 \geq f(S) - c(S)/2$ finally. □

Proof of Theorem 2.1.2. To see $\leq$, let m be a maximum matching. Then, by increasing m by 2 on at most $f(S) - r(m)$ elements of S, we obtain a cover c. Thus, $\varrho(f) \leq m(S)/2 + f(S) - r(m) = f(S) - \nu(f)$. The other direction follows from Claim 2.1.3. □

The *matroid matching problem* (a.k.a. *matroid parity* or *polymatroid parity problem*) is to determine $\nu(f)$ (equivalently $\varrho(f)$) in the sense of good characterization, as well as in an algorithmic sense. For $T \subseteq S$ we can ask also for the more general quantity

$$\delta_T(f) = \min\{|\{s \in S : x_s \not\equiv \chi_T(s)\}| : x \in \mathcal{B}(f) \text{ integer}\},$$

where, by $\equiv$ we always mean the coordinate-wise congruency relation modulo 2. As $\mathcal{B}(f + \chi_T) = \mathcal{B}(f) + \chi_T$, we have $\delta_T(f) = \delta_\emptyset(f + \chi_T)$, $2\nu(f) = f(S) - \delta_\emptyset(f)$, and therefore $2\varrho(f) = f(S) + \delta_\emptyset(f)$. Hence the good characterizations to $\nu(f)$ and $\varrho(f)$ imply each other and the good characterization to $\delta_T(f)$ is equivalent to a good characterization of the maximum matching of a translation of f.

The computation of $\nu(f)$ is hard, more precisely it may need an exponential number of oracle calls if f is given by a value giving oracle (Jensen and Korte [26] and Lovász [35]).

Theorem 2.1.4. *If a 2-polymatroid function $f : 2^S \to \mathbb{Z}_+$ is given by the oracle giving the values of f, then at least $2^{\Omega(|S|)}$ oracle calls are needed to determine $\nu(f)$,*

Proof. Let $\nu \in \mathbb{Z}_+$, and let us define $f : 2^S \to \mathbb{Z}_+$ by

$$f(U) = \begin{cases} 2|U|, & \text{if } |U| < \nu, \\ 2\nu - 1 \text{ or } 2\nu, & \text{if } |U| = \nu, \\ 2\nu, & \text{if } |U| > \nu. \end{cases}$$

It can be checked, that these are all 2-polymatroid functions. Let us assume that one has an algorithm computing $\nu(f)$. Then it has to ask $f(U)$ for all the subsets of S of size ν, otherwise it cannot determine whether $\nu(f) = \nu - 1$ or $\nu(f) = \nu$. If $\nu \approx S/2$, then $2^\nu = 2^{\Omega(|S|)}$. □

Classical NP-complete problems also can be formulated as matroid matching. We can rid out the oracle call so that computing the maximum size clique of graphs becomes a special case. Let $G = (V, E)$ be an undirected graph, and let us modify the above construction as follows. If $X \subseteq V$, $|X| = \nu$, then let $f(X) = 2\nu$ if X is a clique, and $f(X) = 2\nu - 1$ otherwise. As the problem of determining the existence of a clique of size ν is NP-complete, so is the matching problem for such 2-polymatroids.

Parity problems may also appear in somewhat different forms in applications. Let us recall some of them.

The first special case is the 2-*polymatroid parity problem*, which is the parity problem of 2-polymatroid functions (as this is suggested by its name). In fact, polymatroid parity can be reduced to 2-polymatroid parity. For this, let $p : 2^A \to \mathbb{Z}_+$ be a pre-2-polymatroid of f, and $\varphi : A \to S$ the homomorhic map with $\varphi(p) = f$. Then, an even vector $m : A \to \mathbb{Z}_+$ is a matching of p if and only if $\varphi(m)$ is a matching of f. Hence, it would be sufficient to develop the theory to this special case, as it was done by Lovász. However, it is more convenient to use the general form in some applications. Moreover, if we follow Lovász' way, then some important notions and phenomena have to be introduced or shown in both frameworks. Consequently, we develop the basics in the technically bit more messy polymatroid parity framework.

Another form is the *matroid parity* problem, which is as follows. Let $\mathcal{M}$ be a matroid on ground-set E and let $A \subseteq \binom{E}{2}$ be a finite set of (not necessarily disjoint) pairs. If $F \subseteq E$ and $M \subseteq A$, then we use the notations $r_{\mathcal{M}}(F \cup M) = r_{\mathcal{M}}(F \cup \bigcup M)$ and $\mathrm{sp}_{\mathcal{M}}(F \cup M) = \mathrm{sp}_{\mathcal{M}}(F \cup \bigcup M)$. We say that $M \subseteq A$ is a *matching* if $r_{\mathcal{M}}(M) = 2|M|$. Then $p : 2^A \to \mathbb{Z}_+$ defined by $p(B) = r_{\mathcal{M}}(\bigcup B)$ is a 2-polymatroid function, and $M \subseteq A$ is a matching if and only if $m : A \to \mathbb{Z}_+$, $m = 2\chi_M$ is a matching of p.

The last problem is the *maximization of smooth submodular function*s. Let $b : 2^S \to \mathbb{Z}_+$ be a submodular function, which is smooth in the sense that $|b(X) - b(Y)| \leq |X \Delta Y|$ if $X, Y \subseteq S$, (where Δ stands for the symmetric difference). While the maximization of submodular functions is NP-hard in general, it can be tractable for smooth submodular functions, if the corresponding matching problem behaves well. For this, $X \mapsto b(X) - b(\emptyset) + |X|$ is a 2-polymatroid function, and the sets maximizing b are exactly the supports of the maximum matchings of b.

2.2 Circuits and flowers

In order to study the structure of even vectors of polymatroids, let us recall the following notions. The 1-deficient vectors are called *flower*s, i.e. $x : S \to \mathbb{Z}_+$ is a flower if $r(x) = x(S) - 1$. The inclusionwise minimal flowers are called *circuit*s, i.e. $c : S \to \mathbb{Z}_+$ is a circuit if $r(c) = c(S) - 1$ and $r(c - \chi_s) = c(S) - 1$ for every

$s \in \mathrm{supp}(c)$. Similarly, the 2-deficient vectors are called *double flowers* and the inclusionwise minimal double flowers are called *double circuits*. More precisely, a vector $w : S \to \mathbb{Z}_+$ is a double circuit if w is 2-deficient, and $w - \chi_s$ is 1-deficient for every $s \in \mathrm{supp}(w)$. If x is a (double) flower and U is the unique inclusionwise minimal set giving equality in (1.2), then by restricting x to U we get the unique (double) circuit $c \leq x$. If x is a (double) flower, then its unique (double) circuit is denoted by $C(x)$. We say that the *double flower* x is *compatible* if for every $s \in \mathrm{supp}(C(x))$, the support of the unique circuit of the flower $C(x) - \chi_s$ does not contain s.

We will see in Theorem 2.3.1 that from the viewpoint of parity the compatible double circuits are particularly interesting. In some cases however we should deal with non-compatible double circuits too, as compatible double circuits can turn into non-compatible ones at certain polymatroid operations. (E.g. at contraction, see the example after Claim 2.6.5.) In the following paragraphs if we talk about a *double circuit of a matroid* $\mathcal{M}$ with ground-set E, then we mean a set $D \subseteq E$, s.t. $r(D) = |D| - 2$, and $r(D - \{e\}) = |D - \{e\}| - 1$ for every $e \in D$. Hence, this is a restriction of the polymatroidal notion, here we are considering only $\{0, 1\}$-valued vectors (the characteristic vectors of the matroidal double circuits). If $D \subseteq E$ is a double circuit, then the dual of $\mathcal{M}|D$ is a matroid of rank 2 without loops, showing that there exists a so called *principal partition* $D = D_1 \dot\cup D_2 \dot\cup \dots \dot\cup D_d$, $d \geq 2$, s.t. the circuits of D are exactly the sets of form $D - D_i$, $1 \leq i \leq d$. We say that D is *non-trivial* if $d \geq 3$, and *trivial* otherwise. A trivial double circuit is simply the direct sum of two circuits. An exact relation between matroidal and polymatroidal double circuits is as follows:

Proposition 2.2.1. *Let $\mathcal{M}$ be a prematroid of f with ground set E, $D \subseteq E$ and $\varphi(\chi_D) = w$. Then D is a double circuit of $\mathcal{M}$ if and only if w is a double circuit of f.*

Let $\mathcal{M}$ be a prematroid of f and w be a double circuit of f such that there is a set $D \subseteq E$ with $\varphi(\chi_D) = w$. By Proposition 2.2.1, D is a double circuit of $\mathcal{M}$, thus it has a principal partition $D = D_1 \dot\cup D_2 \dot\cup \dots \dot\cup D_{d'}$. We define the principal partition of w as follows. Due to the structure of prematroids it is easy to check that $\mathrm{supp}(w)$ has a partition $U_0 \dot\cup U_1 \dot\cup U_2 \dot\cup \dots \dot\cup U_d$ (called also principal partition) with the property that each set D_j is either a singleton belonging to some E_s with

$w_s \geq 2$ and $s \in U_0$, or is equal to $D \cap \bigcup_{s\in U_h} E_s$ for some $1 \leq h \leq d$. Note that a partition $U_0 \dot\cup U_1 \dot\cup \ldots \dot\cup U_d$ of $\operatorname{supp}(w)$ is the principal partition of w if and only if $w - \chi_s$ is a circuit of f and $w_s \geq 2$ whenever $s \in U_0$, moreover, $w|_{U-U_i}$ is a circuit of f for each $1 \leq i \leq d$. Hence, the double circuit w is *compatible* if $U_0 = \emptyset$. The number d is called the *degree* of the compatible double circuit. The compatible double flower is *non-trivial (NTCDF)* if the degree of its double circuit is at least 3. Hence, $w : S \to \mathbb{Z}_+$ is a compatible double circuit if and only if $\operatorname{supp}(w)$ has a partition $U_1, U_2, \ldots, U_d$, $d \geq 2$ called *principal partition*, s.t. if $x \leq w$, then

$$r(x) = \begin{cases} x(S) - 2, & \text{if } x = w, \\ x(S) - 1, & \text{if } x(U_i) = w(U_i) \text{ for exactly } d-1 \text{ of the } U_i\text{'s}, \\ x(S), & \text{if } x(U_i) = w(U_i) \text{ for at most } d-2 \text{ of the } U_i\text{'s}. \end{cases}$$

Given a double circuit w, an independent vector can be reached either by decreasing w by 2 on an entry, or by 1 on two different entries. If an independent vector can be reached the first way, then w is not compatible; while if we have only the second possibility, then w is compatible. This may be the most expressive view of compatibility. An important consequence is the following:

Corollary 2.2.2. *If $x : S \to \mathbb{Z}_+$ is an even double flower, then either there exists $s \in \operatorname{supp}(x)$ s.t. $x - 2\chi_s$ is a matching and x is non-compatible, or x is a compatible double flower.*

2.3 Structure of even vectors in polymatroids

Let $\nu = \nu(f)$, and let H be the hypergraph with vertex-set S and hyperedge-set $E = \{\operatorname{supp}(C(x)) : x \text{ is an even flower with } r(x) = 2\nu + 1\}$. This section presents the theorem known as Lovász' structure theorem on polymatroid matchings. Of course we formalize it in terms of polymatroids.

Theorem 2.3.1 (Lovász)**.** *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function. Then, at least one of the following possibilities holds.*

(2.1i) $f(S) = 2\nu + 1$.

(2.1ii) There exist a partition $S = S_1 \dot\cup S_2$, $S_i \neq \emptyset$ s.t. $\nu = \nu(f|_{2^{S_1}}) + \nu(f|_{2^{S_2}})$.

(2.1iii) There exists $s \in S$ s.t. $f(s) \geq 2$ and $s \in \mathrm{sp}(m)$ for each maximum matching m of f.

(2.1iv) f has a non-trivial compatible even double flower x with $x(S) = 2\nu + 2$.

Proof. *Case 1.* There exists $s \in S$ with $f(s) < 2$. Then (2.1ii) holds for $S_1 = \{s\}$ and $S_2 = S - S_1$.

Case 2. $f(s) \geq 2$ for every $s \in S$.

Case 2.1. There exists $s \in S$ s.t. $s \in \mathrm{sp}(m)$ for each maximum matching m of f, i.e. (2.1iii) holds.

Case 2.2. For each $s \in S$, there exists a maximum matching m s.t. $s \notin \mathrm{sp}(m)$.

Claim 2.3.2. *Then, $\bigcup E = S$.*

Proof. Let us choose $s \in S$. As (2.1iii) does not hold, there exists a maximum matching m s.t. $r(m + 2\chi_s) \geq m(S) + 1$. But m is a maximum matching, we cannot have $r(m + 2\chi_s) \geq m(S) + 2$. Therefore, $m + 2\chi_s$ is a flower, with $s \in \mathrm{supp}(C(m + 2\chi_s))$. □

Claim 2.3.3. *If $S = S_1 \dot\cup S_2$, $S_i \neq \emptyset$ s.t. each hyperedge of H is contained either by S_1 or by S_2, then (2.1ii) holds for this choice of S_1 and S_2.*

Proof. Let m be a maximum matching. If $m(S_i) \geq 2\nu(f|_{2^{S_i}})$ for $i \in \{1, 2\}$, then we are done. Thus, let m_1 be a maximum matching of $f|_{2^{S_1}}$ and $m_1(S_1) > m(S_1)$. Choose m_1 and m s.t. $(m_1 \wedge m)(S)$ is maximum.

Then, for every $s \in S_1$, either $s \in \mathrm{sp}(m)$ or $m_1(s) \leq m(s)$. Otherwise, let $s \notin \mathrm{sp}(m)$ s.t. $m_1(s) > m(s)$. Hence $m + 2\chi_s$ is a flower with circuit c, $s \in \mathrm{supp}(c)$. As $c \not\leq m_1$ and $c(s) \leq m(s) + 2 \leq m_1(s)$, there exists an $u \in S - \{s\}$ s.t. $c(u) > m_1(u)$. Then $m' = m + 2\chi_s - 2\chi_u$ is a matching having $(m_1 \wedge m)(S) < (m_1 \wedge m')(S)$.

As $m_1(s) \leq m(s)$ for each $s \in S_1 - \mathrm{sp}(m)$, we have $\mathrm{sp}(m_1) \subseteq S - \mathrm{sp}(m)$, hence $m_1 + 2\chi_s$ and $m + 2\chi_s$ are flowers. Let us fix $s \in S_1 - \mathrm{sp}(m)$. Then, we have $C(m_1 + 2\chi_s) = C(m + 2\chi_s)$. Otherwise, let $x, y \in S_1$ s.t. $C(m_1 + 2\chi_s)(x) > C(m+2\chi_s)(x)$ and $C(m+2\chi_s)(y) > C(m_1+2\chi_s)(y)$. Replacing m by $m+2\chi_s-2\chi_y$, and m_1 by $m_1 + 2\chi_s - 2\chi_x$ increases $(m_1 \wedge m)(S)$.

Let $s \in S$ s.t. $m_1(s) > m(s)$. Then there exists a maximum matching n s.t. $s \notin \mathrm{sp}(n)$. Let us choose m_1, m, and n so as to maximize $(m \wedge n)(S)$ (under the primary condition that $(m_1 \wedge m)(S)$ is maximum).

Since $m_1(s) > m(s)$, we have $s \in \mathrm{sp}(m)$. We claim there exists a $z \notin \mathrm{sp}(m)$ with $n(z) > m(z)$. Otherwise, we have $n(S-\mathrm{sp}(m)) \leq m(S-\mathrm{sp}(m))$, $n(\mathrm{sp}(m)) < f(\mathrm{sp}(m))$ as $s \in \mathrm{sp}(m) - \mathrm{sp}(n)$, and $f(\mathrm{sp}(m)) = m(\mathrm{sp}(m))$. The sum of the last three inequalities gives $n(S) < m(S)$ which is a contradiction.

If $z \in S_2$, then $\mathrm{supp}(C(m+2\chi_z)) \cap S_1 = \emptyset$. Choose x s.t. $C(m+2\chi_z)(x) > n(x)$. As $n(z) > m(z)$, $z \neq x$. Then replacing m by $m + 2\chi_z - 2\chi_x$ maintains $m_1 \wedge m$ but increases $(m \wedge n)(S)$.

If $z \in S_1$, then we recall $C(m + 2\chi_z) = C(m_1 + 2\chi_z)$. Choose x s.t. $C(m + 2\chi_z)(x) > n(x)$. Again, $z \neq x$. Then replacing m_1 by $m_1 + 2\chi_z - 2\chi_x$ and m by $m + 2\chi_z - 2\chi_x$ maintains $(m_1 \wedge m)(S)$ but increases $(m \wedge n)(S)$.

□

Claim 2.3.4. *If H is connected, then (2.1i) or (2.1iv) holds.*

Proof. Two circuits c_1 and c_2 are said to be *far* if $c_i = C(m+2\chi_{s_i})$ for a maximum matching m s.t. $r(m+2\chi_{s_1}+2\chi_{s_2}) = 2\nu+2$. Therefore, $s_i \notin \mathrm{supp}(C(m+2\chi_{s_{3-i}}))$.

If $f(S) \leq 2\nu + 1$, then (2.1i) holds and we are done, hence far circuits $C(m + 2\chi_{s_1})$ and $C(m + 2\chi_{s_2})$ exist. Let us choose m, s_1, and s_2 s.t. the distance of the hyperedges $\mathrm{supp}(C(m + 2\chi_{s_1}))$ and $\mathrm{supp}(C(m + 2\chi_{s_2}))$ in the line-graph of H is as small as possible.

Case 1. Suppose first that $\mathrm{supp}(C(m + 2\chi_{s_1})) \cap \mathrm{supp}(C(m + 2\chi_{s_2})) \neq \emptyset$. As $r(m+2\chi_{s_1}+2\chi_{s_2}) = 2\nu+2$, then by Corollary 2.2.2, either $m+2\chi_{s_1}+2\chi_{s_2}-2\chi_u$ is a matching for some $u \in \mathrm{supp}(m+2\chi_{s_1}+2\chi_{s_2})$ or $m+2\chi_{s_1}+2\chi_{s_2}$ is a double-flower with $r(m + 2\chi_{s_1} + 2\chi_{s_2}) = 2\nu + 2$. In the latter case, as $m + 2\chi_{s_1} + 2\chi_{s_2}$ contains two different circuits with intersecting supports, the double-flower is compatible and non-trivial.

Case 2. Next, $\mathrm{supp}(C(m + 2\chi_{s_1})) \cap \mathrm{supp}(C(m + 2\chi_{s_2})) = \emptyset$. Then, there is an intermediate circuit $C(m' + 2\chi_x)$ on a shortest path between $C(m + 2\chi_{s_1})$ and $C(m + 2\chi_{s_2})$. Let us choose m' so as to maximize $(m' \wedge (m + 2\chi_{s_1} + 2\chi_{s_2}))(S)$. As $r(m+2\chi_{s_1}+2\chi_{s_2}) > r(m'+2\chi_x)$, then there exists $y \notin \mathrm{sp}(m'+2\chi_x)$ s.t. $(m+2\chi_{s_1}+2\chi_{s_2})(y) > (m'+2\chi_x)(y)$. Thus, $C(m'+2\chi_x)$ and $C(m'+2\chi_y)$ are far circuits and $\mathrm{supp}(C(m' + 2\chi_x)) \cap \mathrm{supp}(C(m' + 2\chi_y)) = \emptyset$. If $C(m' + 2\chi_y) \leq m + 2\chi_{s_1} + 2\chi_{s_2}$, then $C(m' + 2\chi_y) = C(m + 2\chi_{s_1})$ or $C(m' + 2\chi_y) = C(m + 2\chi_{s_2})$ (as these are the only circuits in $m + 2\chi_{s_1} + 2\chi_{s_2}$), therefore $C(m' + 2\chi_x)$ and $C(m + 2\chi_{s_1})$, or $C(m' + 2\chi_x)$ and $C(m + 2\chi_{s_2})$ would be far with smaller distance. Hence,

$C(m' + 2\chi_y) \not\leq m + 2\chi_{s_1} + 2\chi_{s_2}$, and we can choose z s.t. $C(m' + 2\chi_y)(z) > (m + 2\chi_{s_1} + 2\chi_{s_2})(z)$. Note that $z \neq y$, by the choice of y. Let $m'' = m' + 2\chi_y - 2\chi_z$. By $\mathrm{supp}(C(m'+2\chi_x)) \cap \mathrm{supp}(C(m'+2\chi_y)) = \emptyset$, we have $C(m'+2\chi_x) = C(m''+2\chi_x)$. Then, $(m'' \wedge (m + 2\chi_{s_1} + 2\chi_{s_2}))(S) > (m' \wedge (m + 2\chi_{s_1} + 2\chi_{s_2}))(S)$.

□

□

2.4 Projections and contractions

Let f again be a polymatroid function. Projections and contractions play an important role in the min-max formulas to $\nu(f)$, these min-max formulas are based on the following estimations:

Claim 2.4.1. *If f' is a 1-projection of f, then $\nu(f') \geq \nu(f) - 1$.*

Proof. Let m be a maximum matching of f. If $r_{f'}(m) = m(S)$, then m is a matching of f'. Otherwise, $r_{f'}(m) = m(S) - 1$, i.e., m is a flower of f'. Then, there exists $s \in \mathrm{supp}(m)$ s.t. $m - 2\chi_s$ is a matching of f'. □

Hence, let f' be a projection of f. Then,

$$\nu(f) \leq f(S) - f'(S) + \nu(f').$$

It is also clear, that if $S_1, S_2, \ldots, S_t$ is a partition of S, then

$$\nu(f) \leq \sum_{j=1}^{t} \nu(f|_{2^{S_j}}). \tag{2.2}$$

Last,

$$\nu(f) \leq \left\lfloor \frac{f(S)}{2} \right\rfloor.$$

These together imply that if $S_1, S_2, \ldots, S_t$ is a partition of S, then

$$\nu(f) \leq \sum_{j=1}^{t} \left\lfloor \frac{f(S_j)}{2} \right\rfloor, \tag{2.3}$$

and

$$\nu(f) \leq f(S) - f'(S) + \sum_{j=1}^{t} \left\lfloor \frac{f'(S_j)}{2} \right\rfloor . \tag{2.4}$$

The point throughout this work is to explore some classes of polymatroids where equality can be obtained in (2.3) or in (2.4). It is not hard to show a polymatroid which does not have equality in (2.3). If $|S| = 3$ and $f(X) = |X| + 1$ for each non-empty set $X \subseteq V$, then $\nu(f) = 1$ while the minimum in the right hand side of (2.3) is 2. However, for the projection $f'(X) = |X|$, we have $\nu(f) = f(S) - f'(S) + \nu(f')$ and we have equality in (2.4) when S is partitioned into singletons. In general, we do not know examples where equality cannot be obtained in (2.4). However, in order to obtain a min-max relation to $\nu(f)$ we have to polynomially encode f' and we have to show that f' is a projection of f. It is unlikely that this task can be done in general, the only hope is that f is from a natural class of polymatroids and the projection is defined by a contraction of a natural extension of f.

We say that the projection f' of f *compresses* the compatible double circuit w if $r_{f'}(c) < r_f(c)$ for every circuit c of w.

Theorem 2.4.2. *Let y be an even NTCDF with $f(y) = 2\nu(f) + 2$. If the 1-projection f' compresses the double circuit of y, then $\nu(f') = \nu(f) - 1$.*

Proof. Suppose for contradiction that m is a matching of f' and $m(S) = 2\nu(f)$. Let moreover $(m \wedge y)(S)$ be as large as possible. As $r_{f'}(y) \geq r_f(y) - 1 = 2\nu(f) + 1 > r_{f'}(m)$, let $s \notin \mathrm{sp}_{f'}(m)$ s.t. $y(s) > m(s)$. Then, we have $2\nu(f) + 1 \geq r_f(m + 2\chi_s) \geq r_{f'}(m + 2\chi_s) \geq 2\nu(f) + 1$, so $m + 2\chi_s$ is a flower of f and also of f'. As $f - f'$ is non-decreasing, $r_f(x) = r_{f'}(x)$ for every $x \leq m + 2\chi_s$. Therefore, $C_f(m + 2\chi_s) = C_{f'}(m + 2\chi_s)$. Let c be this circuit. If $c \not\leq y$, then let u s.t. $c(u) > y(u)$. Then $m' = m + 2\chi_s - 2\chi_u$ is a matching of f s.t. $(m' \wedge y)(S) > (m \wedge y)(S)$. Thus, $c \leq y$. But this contradicts the fact that the 1-projection compresses the circuits of the flower y. $\square$

2.5 Linear polymatroids and modular lattice of flats

All the matroid parity problems which were solved in the seventies were reduced to the matching problem of graphs or to the matroid intersection problem. The very first case which is unlikely to be reduced to these two easier problems and happened to be solvable, is the parity problem of linear polymatroids.

Let L be a *full linear space* and let A be a finite set of its subspaces. Then the dimension is a natural polymatroid function on the subsets of $S = L \cup A$. Lovász' theorem on the matching problem of linear matroids is the following:

Theorem 2.5.1 (Lovász)**.** *Let A be finite set of subspaces of a linear space L and let $S = A \cup L$. Then,*

$$\nu(\dim|_{2^A}) = \min\left(\dim(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(\dim/Z)(A_j)}{2} \right\rfloor\right). \tag{2.5}$$

where the minimum is taken for all subspaces Z of L and for all partitions $A_1, A_2, \ldots, A_t$ of A.

The matroids considered in daily life are unlikely to be non-linear, thus Lovász' work have useful consequences in both theory and practice. Interesting engineering applications are the problem of pinning down a minimum number of vertices of a planar bar-and-joint framework to obtain a rigid framework or the unique solvability problem of electrical networks containing gyrators. For other combinatorial applications see [38; 49].

We have to note that it is not immediately clear that (2.5) gives a good characterization. If the vector space is over the rational field, then the numbers defining Z can be too large as compared to the size of the definition of the original problem. This is not a problem for small finite fields and in fact can be solved also for the rational field, see [45; 46; 56].

The only property of full linear matroids used in the proof of Theorem 2.5.1 is the modular structure of the lattice of flats of the polymatroid function $\dim : 2^S \to \mathbb{Z}_+$. We say that a polymatroid has a *modular lattice of flats* if

$$f(X) + f(Y) = f(X \cap Y) + f(X \cup Y),$$

for any two flats X, Y. It is clear that full linear polymatroids have modular lattice of flats. As we suggested, Theorem 2.5.1 can be generalized to polymatroids with modular lattice of flats:

Theorem 2.5.2. *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function with modular lattice of flats. Then, for any $A \subseteq S$,*

$$\nu(f|_{2^A}) = \min\left(f(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(f/Z)(A_j)}{2} \right\rfloor\right). \tag{2.6}$$

where the minimum is taken for all $Z \subseteq S$ and for all partitions $A_1, A_2, \ldots, A_t$ of A.

Another notable example for polymatroids with modular lattice of flats arises in the matching problem of graphs. Now, let $G = (V, E)$ be an undirected graph, and let $q' : 2^{V \cup E} \to \mathbb{Z}_+$ be the 2-polymatroid function with $q'(U) = |(U \cap V) \cup \bigcup(U \cap E)|$. Then, q' has modular lattice of flats, as it arises from the free matroid on V by adding some flats of rank 2. Moreover, $q'|_{2^E} = q$, where q is the 2-polymatroid function defined in Section 2.1. Therefore, the maximum matching of G has $\nu(q'|_{2^E})$ edges, which has the characterization as in Theorem 2.5.2. After some technical simplifications, it gives back the Berge-Tutte formula.

Theorem 2.5.3 (Berge and Tutte, [1])**.** *Let $G = (V, E)$ be an undirected graph. Then the maximum matching of G has cardinality*

$$\min_{X \subseteq V}\left(|X| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C|}{2} \right\rfloor\right), \tag{2.7}$$

where $\mathcal{C}$ denotes the set of vertex-sets of the components of $G[V - X]$.

Proof. First, we have

$$\nu(q'|_{2^E}) = |X| + \sum \left\lfloor \frac{|X \cup \bigcup E_i| - |X|}{2} \right\rfloor$$

for some $X \subseteq V$ and a partition $E_1, E_2, \ldots, E_t$ of E. Then, by choosing t to be minimal, we have $(X \cup \bigcup E_i) \cap (X \cup \bigcup E_j) = X$ if $i \neq j$, and there is no edge between $(\bigcup E_i) - X$ and $(\bigcup E_j) - X$. Finally,

$$|X| + \sum \left\lfloor \frac{|X \cup \bigcup E_i| - |X|}{2} \right\rfloor \geq |X| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C|}{2} \right\rfloor.$$

□

2.6 The double circuit property

Suppose that we have a polymatroid coming from a combinatorial application. It is a common experience that these polymatroids happen to be non-linear once in a blue moon. Though linearity can be clear from the definition of the polymatroid, we still might not be aware of a deterministic polynomial algorithm representing the polymatroid. Having an explicit representation is indispensable either for defining Z in (2.5) or to run an algorithm which computes a maximum matching and a dual certificate proving that equality holds in (2.5). The most simple examples are the transversal matroid and the 2-dimensional generic rigidity matroid; the phenomenon elaborated more thoroughly in Chapter 3. However, we are not finished with having a linear representation. Even if it is at hand, the obtained min-max relation (2.5) will not reflect the combinatorial behavior of the original polymatroid, since by the embedding into a full linear space the contraction in the dual side of (2.5) hardly has any combinatorial meaning.

A starting point to address this problem is an abstract generalization of Theorem 2.5.1 by Dress and Lovász [10]. Their motivation was to handle more general matroids then linear ones. We use their result in a completely different direction. Our polymatroids are linear (except those presented in Chapter 7) but not full ones. Hence, we can keep the combinatorial structure at contractions and we will be able to explore some graph theoretical applications. Dress and Lovász observed that the tractability of the linear and some simple combinatorial special cases is due to a common abstract property which is more general than modularity. Reformulated in polymatroidal terms, they say that the polymatroid function f has the *double circuit property* (*DCP* for short) if

$$(f/z)\left(\bigcap_{i=1}^{d} \mathrm{sp}_{f/z}(x_i)\right) > 0 \tag{2.8}$$

holds for each NTCDC x of each contraction f/z of f, where $x_1, x_2, \ldots, x_d$ are the circuits of x. Then, we get:

Theorem 2.6.1. *If f is a polymatroid function which has the DCP and $A \subseteq S$, then*

$$\nu(f|_{2^A}) = \min\left(f(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(f/Z)(A_j)}{2} \right\rfloor\right),$$

where the minimum is taken for all $Z \subseteq S$ and for all partitions $A_1, A_2, \dots, A_t$ of A.

Proof. First, we prove that there exists a vector $z : 2^S \to \mathbb{Z}_+$ and a partition $A_1, A_2, \dots, A_t$ s.t.

$$\nu(f|_{2^A}) = r_f(z) + \sum_{j=1}^{t} \left\lfloor \frac{(f/z)(A_j)}{2} \right\rfloor . \tag{2.9}$$

Let $z \in \mathbb{Z}_+^S$ be a vector which gives equality in $\nu(f|_{2^A}) = r_f(z) + \nu((f/z)|_{2^A})$ and maximizes $r_f(z)$ among these vectors. Moreover, let $A_1, A_2, \dots, A_t$ a partition of A s.t. $\nu((f/z)|_{2^A}) = \sum_{j=1}^{t} \nu((f/z)|_{2^{A_j}})$, and in addition, let t be as big as possible. We claim that $\nu((f/z)|_{2^{A_j}}) = \left\lfloor \frac{(f/z)(A_j)}{2} \right\rfloor$. Let us apply Theorem 2.3.1 to $(f/z)|_{2^{A_j}}$. In case (2.1i), we are done, while in case (2.1ii), there would be a partition with larger t. In case (2.1iii) we replace z by $z + \chi_s$. Then, $r_f(z + \chi_s) = r_f(z) + 1$, $\nu((f/z)|_{2^{A_j}})$ drops by 1 and the other $\nu((f/z)|_{2^{A_i}})$'s do not increase, which contradicts the choice of z. For case (2.1iv), let $s \notin \mathrm{sp}_f(z)$ be an element which is in the span of each circuit of the NTCDC. Then, by replacing z by $z + \chi_s$, $\nu((f/z)|_{2^{A_j}})$ drops by 1, the other $\nu((f/z)|_{2^{A_i}})$'s do not change, which again contradicts the choice of z. Hence, we have (2.9).

Next, we prove that $\mathrm{supp}(z) \subseteq \mathrm{sp}(z)$. Let us choose z and $A_1, A_2, \dots, A_t$ so that fist, $z(\mathrm{supp}(z) - \mathrm{sp}(z))$ is as small as possible, and second, t is as small as possible under the primary condition. Then, $\mathrm{sp}_{f/z}(A_i) \cap \mathrm{sp}_{f/z}(A_j) = \mathrm{sp}_f(z)$ for every $1 \le i < j \le t$. Otherwise, we could delete the partition members A_i and A_j and introduce a new member $A_i \cup A_j$. Therefore, if $s \in \mathrm{supp}(z) - \mathrm{sp}(z)$, then s is contained by at most one of the $\mathrm{sp}_{f/z}(A_i)$'s, and we could replace z by $z - \chi_s$. $\square$

It is a frequent way of defining polymatroids in combinatorial applications as homomorphic maps of matroids. The matching problem of these polymatroids are tractable if the underlying matroids are DCP:

Theorem 2.6.2. *Let $\mathcal{M}$ be a matroid with ground-set E which has DCP. Let moreover $A \subseteq 2^E$, and $f : 2^{E \cup A} \to \mathbb{Z}_+$, $f(F \cup B) = r_{\mathcal{M}}(F \cup \bigcup B)$ for $F \subseteq E$ and $B \subseteq A$. Then, we have*

$$\nu(f|_{2^A}) = \min \left(f(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(f/Z)(A_j)}{2} \right\rfloor \right), \tag{2.10}$$

where the minimum is taken for all $Z \subseteq E$ and for all partitions $A_1, A_2, \dots, A_t$ of A.

Proof. Setting $A' = \left\{\binom{A_i}{2} : A_i \in A\right\}$, it is natural to consider the matching problem of the 2-polymatroid $f' : 2^{E \cup A'} \to \mathbb{Z}_+$, the values of f' are inherited again from $r_{\mathcal{M}}$. Then, $\nu(f|_{2^A}) = \nu(f|_{2^{A'}})$. First, it is not hard to see that the DCP of $\mathcal{M}$ implies that for any even NTCDC x of f'/Z, $Z \subseteq E$ with $\operatorname{supp}(x) \subseteq A'$ we have

$$(f'/Z)\left(\bigcap_{i=1}^{d} \operatorname{sp}_{f'/Z}(x_i)\right) > 0, \tag{2.11}$$

see e.g. [51]. Then, $\nu(f'|_{2^{A'}}) = f'(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(f'/Z)(A'_j)}{2} \right\rfloor$ for some $Z \subseteq E$ and a partition $A'_1, A'_2, \dots, A'_t$ of A'. It is not hard to see, that then $\nu(f|_{2^A}) = f(Z) + \sum_{j=1}^{t'} \left\lfloor \frac{(f/Z)(A_j)}{2} \right\rfloor$ for some $Z \subseteq E$ and a partition $A_1, A_2, \dots, A_{t'}$ of A. $\square$

Now we turn to a detailed examination of double circuits. If we consider a NTCDC x with principal partition $U_1, U_2, \dots, U_d$, then we tacitly assume that the circuits are denoted by $x_1, x_2, \dots, x_d$, where

$$x_i(s) = \begin{cases} x(s), & \text{if } s \in U_j,\ j \neq i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Claim 2.6.3. *Let x be a NTCDC of the polymatroid function f.*

(2.13i) If $i \in T \subseteq [d]$, then

$$f\left(\bigcap_{t \in T} \operatorname{sp}(x_t)\right) \le f\left(\bigcap_{t \in T - \{i\}} \operatorname{sp}(x_t)\right) - x(U_i) + 1. \tag{2.14}$$

(2.13ii) If $T \subseteq [d]$, then

$$f\left(\bigcap_{t \in T} \operatorname{sp}_f(x_t)\right) \le \sum_{t \in [d] - T} x(U_t) + |T| - 2. \tag{2.15}$$

(For notational convenience we assume $\bigcap_{t \in \emptyset} \operatorname{sp}(x_t) = \operatorname{sp}(x)$.)

Proof. For (2.13i), by submodularity,

$$f\left(\bigcap_{t\in T}\operatorname{sp}(x_t)\right)+f\left(\operatorname{sp}(x_i)\cup\bigcap_{t\in T-\{i\}}\operatorname{sp}(x_t)\right)\le f\left(\bigcap_{t\in T-\{i\}}\operatorname{sp}(x_t)\right)+f\left(\operatorname{sp}(x_i)\right).$$

The first statement follows from $f\left(\operatorname{sp}(x_i)\cup\bigcap_{t\in T-\{i\}}\operatorname{sp}(x_t)\right)=f(\operatorname{sp}(x))$ for which we use $\operatorname{supp}(x)\subseteq\operatorname{sp}(x_i)\cup\bigcap_{t\in T-\{i\}}\operatorname{sp}(x_t)\subseteq\operatorname{sp}(x)$. If $|T|\le 1$, then (2.15) holds. Otherwise, it follows from (2.13i) by induction on $|T|$. □

Hence, if we need $f\left(\bigcap_{t\in T}\operatorname{sp}(x_t)\right)$ to be large, then the most what we can obtain is equality in (2.15). If the lattice of flats is modular, then equality holds in (2.15). This proves Theorem 2.5.2 and Theorem 2.5.1.

Theorem 2.6.4. *If $f:2^S\to\mathbb{Z}_+$ is a polymatroid function with modular lattice of flats, then it has the DCP.*

Proof. First, it is clear that contractions of polymatroids with modular lattice of flats have again modular lattice of flats, by the definition of contraction. The proof is obtained by following the proof of Claim 2.6.3. By the modularity, we have equality in (2.14). Then, we also have equality everywhere in the second part, as well as in (2.15). □

The definition of the DCP specifies a condition for each contraction of the polymatroid. If the examined class of polymatroids is not closed under taking contractions, then this definition is hard to use. However, there is a simple connection between the double circuits of the contractions and of the original polymatroid:

Claim 2.6.5. *Let $z\in S$, $f(z)>0$ and let x be a NTCDC of f/χ_z with principal partition $U_1,U_2,\ldots,U_{d'}$. If $z\notin\operatorname{supp}(x)$, then either*

(2.16i) x is a compatible double circuit of f with the same principal partition, or

(2.16ii) $x+\chi_z$ is a compatible double circuit of f with principal partition $\{z\},U_1,\ldots,U_{d'}$, or

(2.16iii) there exists $1\le j\le d'$ s.t. $x+\chi_z$ is a compatible double circuit of f with principal partition $U_1,\ldots,U_{j-1},U_j\cup\{z\},U_{j+1},\ldots,U_{d'}$.

Proof. Clearly, $r_f(x+\chi_z) = (x+\chi_z)(S) - 2$. If $r_f(x) = x(S) - 2$, then we are at (2.16i). Otherwise, $r_f(x) = x(S) - 1$ and $r_f(x+\chi_z-\chi_w) = x(S) - 1$ for every $w \in \mathrm{supp}(x)$, hence, $x+\chi_z$ is a compatible double circuit of f. For each circuit c of f/χ_z either c or $c+\chi_z$ is a circuit of f, so we are done. □

If $z \in \mathrm{supp}(x)$, then the statement of Claim 2.6.5 is not true anymore, i.e. there are NTCDCs of the contraction which do not correspond to NTCDCs of the original polymatroid this way. For this, let $S = \{v_1, v_2, v_3, v_4\}$, $f(U) = 2$ if $|U| = 1$, $f(U) = 2|U|$ if $v_1 \notin U$, and $f(U) = 2 + |U|$ if $\{v_1\} \subsetneq U$. Then, f is a 2-polymatroid function, and $x = (1, 2, 2, 2)$ is a NTCDC of f/χ_{v_1}. But f has no NTCDCs, as v_1 is contained in the support of any circuit with support of size at least 2. In fact, $x + \chi_{v_1}$ is a non-compatible double circuit of f. Never mind, we got an important corollary:

Corollary 2.6.6. *Let $\mathcal{M}$ be a prematroid of the polymatroid function f. Then, $\mathcal{M}$ has the DCP if and only if f has the DCP.*

In their original work, Dress and Lovász define the DCP in a slightly stronger way. They say the DCP (and only for matroids) if

$$f\left(\bigcap_{1\le i\le d} \mathrm{sp}_f(x_i)\right) \ge d - 2$$

holds for each NTCDC x of f. We call this *modular double circuit property* (*MDCP*). It should be mentioned here that the auxiliary adjective "modular" stands for indicating the modular structure of NTCDCs:

Claim 2.6.7. *Let f be a polymatroid function. Then, f has the MDCP if and only if (2.15) holds with equality for each NTCDC.*

Proof. If f has the MDCP, then we must have equality everywhere in the proof of Claim 2.6.3. □

This intermediate property, the MDCP, helps to prove the DCP in some cases:

Claim 2.6.8. *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function having the MDCP. If $z \in \mathbb{Z}_+^S$ with $z \le \mathbf{1}$, then*

$$(f/z)\left(\bigcap_{1\le i\le d'} \mathrm{sp}_{f/z}(x'_i)\right) \ge d' - 2$$

for each NTCDC x' of f/z with circuits $x'_1, x'_2, \ldots, x'_{d'}$ s.t. $\operatorname{supp}(x')$ *and* $\operatorname{supp}(z)$ *are disjoint.*

This is particularly useful if we restrict ourselves to matroids:

Corollary 2.6.9. *The class of matroids having the MDCP is closed under taking contractions. If the matroid $\mathcal{M}$ has the MDCP, then it has the DCP.*

Proof of Claim 2.6.8. By induction, it is sufficient to prove the statement for $z = \chi_s$, $f(s) > 0$. Let therefore x be a NTCDC of f/z with $\operatorname{supp}(x) \subseteq S - \operatorname{supp}(z)$ with principal partition $U_1, U_2, \ldots, U_d$ and with circuits $x_1, x_2, \ldots, x_d$. Now we can use Claim 2.6.5. In case (2.16i) we are done. Let us assume (2.16ii). As, $x_i + z$ is a circuit of f for $1 \leq i \leq d$, we have $\operatorname{sp}_f(x_i) = \operatorname{sp}_f(x_i + z)$. Therefore,

$$f\left(\bigcap_{i=1}^{d} \operatorname{sp}_f(x_i)\right) \geq f\left(\operatorname{sp}_f(x) \cap \bigcap_{i=1}^{d} \operatorname{sp}_f(x_i + z)\right) \geq (d+1) - 2,$$

which gives

$$(f/z)\left(\bigcap_{i=1}^{d} \operatorname{sp}_{f/z}(x_i)\right) \geq d - 2.$$

Last, in case (2.16iii), let $j = 1$.

$$f\left(\operatorname{sp}_f(x_1) \cap \bigcap_{2 \leq i \leq d} \operatorname{sp}_f(x_i + z)\right) \geq d - 2.$$

If we prove that $s \notin \operatorname{sp}_f(x_1) \cap \bigcap_{2 \leq i \leq d} \operatorname{sp}_f(x_i + z)$, then we are done by

$$(f/z)\left(\bigcap_{1 \leq i \leq d} \operatorname{sp}_{f/z}(x_i)\right) \geq (f/z)\left(\operatorname{sp}_f(x_1) \bigcap_{2 \leq i \leq d} \operatorname{sp}_f(x_i)\right) = f\left(\operatorname{sp}_f(x_1) \bigcap_{2 \leq i \leq d} \operatorname{sp}_f(x_i)\right).$$

If $s \in \operatorname{sp}_f(x_1)$, then $x_1(S) - 1 = r_f(x_1) = r_f(x_1 + z)$, and

$$\begin{aligned} x_1(S) - 1 + x_d(S) = r_f(x_1 + z) + r_f(x_d + z) \geq \\ r_f((x_1 \wedge x_d) + z) + r_f(x + z) = ((x_1 \wedge x_d) + z)(S) + (x + z)(S) - 2 \end{aligned}$$

by submodularity, which is a contradiction. $\square$

The converse of Corollary 2.6.9 might not be true, i.e. there can be DCP matroids which are not MDCP. It would be nice to see an example.

The relation of MDCP and DCP of matroids, polymatroids and their prematroids is summarized as follows:

$$\begin{array}{ccccl}
\text{DCP} & \overset{?}{\Longrightarrow} \atop \not\Longleftarrow & \text{MDCP} & & \Big\} \text{ polymatroids} \\
\Updownarrow & & \Uparrow\not\Downarrow & & \quad \wr \text{ forming a prematroid} \\
\text{DCP} & \overset{?}{\Longrightarrow} \atop \Longleftarrow & \text{MDCP} & & \Big\} \text{ matroids}
\end{array}$$

In most of our cases, the polymatroid function $\widehat{b}$ is constructed from an intersecting submodular function b, as in (1.3xiii). In this case, the optimum of

$$\min\left(\widehat{b}(Z)+\sum_{j=1}^{t}\left\lfloor\frac{(\widehat{b}/Z)(A_j)}{2}\right\rfloor\right)$$

is attained in a special form. It equals

$$\min\left(\widehat{b}(Z)+\sum_{j=1}^{t}\left\lfloor\frac{b_Z(U_j)}{2}\right\rfloor\right),$$

where the minimum is taken for every $Z \subseteq S$ and for every family of sets $U_1, U_2, \dots, U_t \in \mathfrak{L}_Z - \{\emptyset\}$ s.t. $A \subseteq \bigcup_{j=1}^{t} U_j$. We may assume that for any $i \neq j$ with $U_i \cap U_j \neq \emptyset$, there exists $U \in \mathfrak{F}_Z$ with $U = U_i \cap U_j$. Moreover, if $U \in \mathfrak{F}_Z$, then there exist $U_i \neq U_j$, $U \subsetneq U_i, U_j$ s.t. $U_i \cap U_j = U$.

2.7 Matroid properties implying the MDCP

Dress and Lovász [10] proved that full members of some natural matroid classes have the MDCP even though they do not have modular lattice of flats.

The matroids they considered are the following. Let L be a field extension of the field K. The subsets of L which are algebraically independent over K are called independent, and this independence defines the *full algebraic matroid* with ground set L. The obtained matroid has the MDCP. Dress and Lovász have shown also that the transversal matroid in which each hyperedge e is a member of the ground set with infinite multiplicity (in fact a large finite multiplicity is sufficient for each hyperedge) (*full transversal matroid*), and also the graphic matroids of finite complete graphs (*full graphic matroid*) have the MDCP.

They introduced the following matroid properties implying the MDCP. Let $\mathcal{M}$ be a matroid with ground set E. The set S is said to be *in series* in U if S is a circuit of $\mathcal{M}/(U-S)$. The matroid $\mathcal{M}$ is said to have the *series reduction property* if for all $S \subseteq U \subseteq E$ s.t. S is in series in U, there is an element $\beta \in E$ s.t. for each $U \subseteq S$, $S \cup T$ is a circuit if and only if $\{\beta\} \cup T$ is a circuit. We say that $\mathcal{M}$ has the *weak series reduction property* if the above holds for each S and U s.t. in addition $U - S$ is connected in $\mathcal{M}$. It is clear that the series reduction property implies the weak one. Dress and Lovász have shown that these properties imply the MDCP, the full linear, and full algebraic matroids have the series reduction property, while full graphic, and full transversal matroids have the weak one.

Björner and Lovász [3] went further, by observing that the so called pseudo-intersections have important role, they introduced the class of *pseudomodular matroids*. One of the several equivalent definitions of pseudomodularity is that for any three flats x, y, z of the matroid s.t. x covers $x \wedge z$ and y covers $y \wedge z$, we have $r(x \wedge y) - r(x \wedge y \wedge z) \leq 1$, where covering and the lattice operations are considered w.r.t. the lattice of flats. Hochstättler and Kern [23] proved that pseudomodular matroids have the MDCP. We are not examining pseudomodularity thoroughly, but we mention and will use some of their important properties [3]. First, uniform matroids and graphic matroids of complete graphs are pseudomodular. Pseudomodularity is closed under taking direct sum (thus partition matroids are pseudomodular). Pseudomodularity is closed under the operation of adding a generic element from a flat (the operation is known as principal extension in lattice theory).

For filling the gap in the hierarchy of DCP matroid classes, Kromberg [28] and Tan [54] proved that matroids having series reduction property are pseudomodular. While Kromberg's proof is short, Tan also proves that even the weak series reduction property implies pseudomodularity.

2.8 Connection with polynomial matrices

The rank computation problem of polynomial matrices includes various fundamental problems in combinatorics. Let $x_1, x_2, \ldots, x_k$ be indeterminates, and let A be a matrix whose entries are polynomials of these indeterminates over the reals. The task is to compute the rank of A. As a polynomial number of elementary operations cannot be performed efficiently with these polynomials, the naive approach of computing the rank by Gaussian elimination does not work.

For simplicity, let A be a square matrix and consider the problem of determining whether A is singular. Clearly, A is non-singular if there are reals $\tilde{x}_i$ s.t. the matrix $\tilde{A}$ obtained from A by substituting the $\tilde{x}_i$ values into the variables x_i is non-singular. If $\tilde{A}$ is non-singular, then it is non-singular for almost all choices of the $\tilde{x}_i$'s. This observation is of algorithmic interest. We can substitute random real values into the indeterminates and we can conclude from the rank of the resulting matrix. The conclusion whether the determinant of A is 0 is not always correct, but the error probability is very small if we choose the random values from large finite ranges (see [53]).

For a detailed study of polynomial matrices see [36] and [38]. We just mention that the intersection problem of linear matroids, determining $\widehat{f}(S)$ for the linear polymatroid $f : 2^S \to \mathbb{Z}_+$, computing the rank of generic rigidity matroids (in any dimension), and even the linear matroid parity problem are special cases of the rank computation problem.

We have mentioned parity problems of linear matroids where the matroid is represented with vectors of polynomials. The prototype example is the 2-dimensional generic rigidity matroid of a graph $G = (V, E)$. Let $a_{uv} = \chi_u - \chi_v \in \mathbb{R}^V$ be the incidence vector of the edge uv (according to a reference orientation), and let $x_v, y_v,\ v \in V$ be a set of indeterminates. By associating the $2|V|$-dimensional vector $q_{uv} = ((y_u - y_v)a_{uv}, (x_u - x_v)a_{uv})$ with the edge uv we get a representation of the 2-dimensional generic rigidity matroid of G.

The 2-polymatroid parity problem can be obtained as follows [38]. Suppose that the 2-polymatroid is represented by the pairs of vectors $(a_s, b_s) \in \mathbb{R}^n \times \mathbb{R}^n$, $s \in S$. Let $x_s,\ s \in S$ be indeterminates again, and let A be the matrix with $a_{i,j} = \sum_{s\in S}((a_s)_i(b_s)_j - (a_s)_j(b_s)_i)$. The maximum number of pairs $M \subseteq S$ s.t. the vectors $\{a_s, b_s : s \in M\}$ are collectively independent is exactly the rank of A.

By putting together the two constructions, we can obtain a rank computation problem of a polynomial matrix which is equivalent to a parity problem to the 2-dimensional generic rigidity matroid with some pairing of the elements. This approach does not give a good characterization, and algorithm is obtained only if randomization is allowed. In the following chapter we show a combinatorial characterization for the parity problem of the 2-dimensional generic rigidity matroid. We show in fact a good characterization for the parity problem of the wider class of count matroids. However, most of the algorithmic questions are left open.

Chapter 3

(k, l)-matroids

One of the most simple matroids defined by intersecting submodular functions having the DCP is the full graphic matroid, i.e. the graphic matroid of a complete graph. It can be seen easily that the only non-trivial double circuits of graphic matroids are the graphs having 3 openly vertex-disjoint paths between 2 different vertices u and v; see Figure 3.1. The paths form the classes of the principal

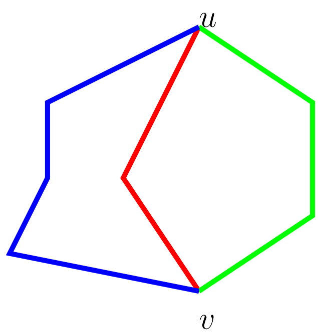

Figure 3.1: A double circuit of degree 3 in $\mathcal{M}_{1,1}$

partition. There is a simple reasoning that these are the only non-trivial double circuits of graphic matroids. If a non-trivial double circuit of a matroid has degree d, then by contracting all but one element from each class of the principal partition, we get a $U_{d,d-2}$ minor, and a $U_{d,2}$ by taking the dual. As the graphic matroid is binary (and has no $U_{4,2}$ minor), it cannot have a non-trivial double circuit of degree bigger than 3. Then we can check that the only double circuits of degree 3 of graphic matroids are as in Figure 3.1.

In order to have the MDCP, an edge between u and v must be in the ground-set. And this is sufficient: we can conclude that the full graphic matroid has the MDCP [10]. It also has the DCP either by Corollary 2.6.9, or by the fact that the class of (full) graphic matroids is closed under taking contractions (up to deleting the loops).

This construction can be generalized in the following natural way. Let $k \geq 1$, $l \geq 0$ be fixed integers and let $H = (V, E)$ be a finite hypergraph. Suppose that each hyperedge is of size at least $\frac{l}{k}$. Let $\mathcal{M}_{k,l}(H)$ be the matroid with ground-set E s.t. $F \subseteq E$ is independent if and only if $|F'| \leq k|\bigcup F'| - l$ for each $\emptyset \neq F' \subseteq F$. It can be checked easily that $\mathcal{M}_{k,l}(H)$ is indeed a matroid, we call it a (k, l)-*matroid*.

Of course, the rank-function $r_{\mathcal{M}_{k,l}(H)}$ can be defined as a Dilworth truncation of an intersecting submodular function (see 1.3xi).

Let $\mathcal{L} = \{\emptyset\} \cup \{\{e\} : e \in E\} \cup \{E[X] : X \subseteq V\}$,

$$(3.1)\quad b(F) \quad = \quad \begin{cases} \min\{k|X| - l : X \subseteq V, F = E[X]\}, & \text{if } |F| \geq 2, \\ \min\{1, \min\{k|X| - l : X \subseteq V, F = E[X]\}\}, & \text{if } |F| = 1, \\ 0, & \text{if } F = \emptyset. \end{cases}$$

Let moreover $E[X_1] \vee E[X_2] = E[X_1 \cup X_2]$ if $E[X_1] \cap E[X_2] \neq \emptyset$, and $F_1 \vee F_2 = F_1 \cup F_2$ if one of F_1 and F_2 is a singleton contained by the other set. Therefore, $\widehat{b}$ is a matroid rank function on E, namely $r_{\mathcal{M}_{k,l}(H)} = \widehat{b}$. The matroids $\mathcal{M}_{k,l}(H)$ might be called count matroid in the literature, we will call them (k, l)-matroids according to the two parameters. Unlike in the case of the graphic matroid, it does not seem to be obvious to characterize the non-trivial double circuits of $\mathcal{M}_{k,l}(H)$. (k, l)-matroids other than the graphic ones are not necessarily binary, (in fact they have various non-trivial double circuits of arbitrarily large degree).

We will see right now that (k, l)-matroids are linear. In this sense they model well our effort that instead of following the linear approach we aim to discover the combinatorial structure of (k, l)-matroids which helps us in the solution of their parity problems. The matroid $\mathcal{M}_{k,l}(H)$ is linear by the following. Let L be the direct sum of k-dimensional real linear spaces L_v $(v \in V)$. Let P be a subspace of $\sum_{v \in V} L_v$ of co-dimension l which is chosen in general position (w.r.t. L_v, $v \in V$). By being in general position w.r.t. L_v, $v \in V$ we mean that all the non-trivial algebraic dependencies are avoided i.e. the coordinates defining P are algebraically independent over the field of rationals extended by the coordinates of L_v, $v \in V$. For $e \in E$, let l_e be a member of $(\sum_{v\in e} L_v) \cap P$, s.t. l_e $(e \in E)$ together are in general position (w.r.t. L_v, $v \in V$ and P). By associating the linear element l_e with e we get a representation of $\mathcal{M}_{k,l}(H)$. For more on the construction, see Lovász [32] and Lovász and Yemini [40]. Here we encounter the situation mentioned in Chapter 2, that we are not able to compute vectors being in general position in an algorithmic way, nor a representation. A parity problem of a (k, l)-matroid therefore can be solved by applying a linear algorithm to a representation using random numbers, or the problem can be reduced to the rank computation of a polynomial matrix as it is sketched in Section 2.8. Both approaches may err with small probability.

For the purely combinatorial approach, first, let $l = 0$. For $k = 1$, let us consider the graph of Figure 3.2. It is a double circuit of degree 3; the classes of the principal partition are formed by the parallel edges. The singleton on v have to be in the ground-set of the matroid, otherwise the DCP does not hold. If H contains only

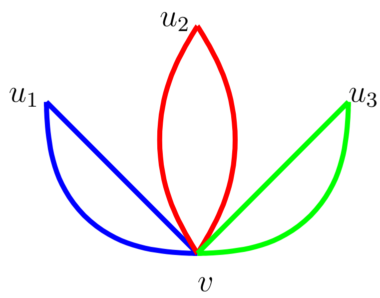

Figure 3.2: A double circuit of degree 3 in $\mathcal{M}_{1,0}$

singletons, then $\mathcal{M}_{k,0}(H)$ is a partition matroid, it has the MDCP, moreover it is pseudomodular. If each singleton of V is in E with multiplicity at least k, then adding a hyperedge $X \neq \emptyset$ to H means that the matroid is extended with a generic element from the flat generated by $E[X]$. As pseudomodularity is closed under taking principal extensions [3], the resulting matroid is pseudomodular. Hence, $\mathcal{M}_{k,0}(H)$ is pseudomodular for any hypergraph H which contains each element of the ground set as a singleton with multiplicity at least k.

We have seen above that the full graphic matroids have the MDCP. If H is a hypergraph and $k = l = 1$, then we speak about a *hypergraphic matroid.*

This matroid (and also its contractions) can have double circuits of arbitrarily large degree. If H contains a hyperedge of size $d-1$ with multiplicity d, then $\mathcal{M}_{1,1}(H)$ is a double circuit of degree d. The best what we can do is that we do not deal with the problem of characterizing the non-trivial double circuits of $\mathcal{M}_{1,1}(H)$. We have to note also, that the class of hypergraphic matroids is not closed under taking contractions anymore. (We do not think that a contraction-closed closure of the hypergraphic matroids should be searched within the class of (k,l)-matroids.) It is a bit more laborious, but it is proved in [3] that full graphic matroids are pseudomodular. Adding a hyperedge X to H means again adding a generic element in the flat generated by $E[X]$. As above, we conclude that $\mathcal{M}_{1,1}(H)$ is pseudomodular if $\binom{V}{2} \subseteq E$.

There is a little chance that similar techniques help to prove the MDCP for larger k's and l's. In what follows, we follow a completely different way. Let $l = ck + d$ where c, d are integers with $0 \leq d < k$. We could think from the above examples that if H contains the sets of size $c+1$ with large multiplicity, then we have the DCP. Even this is false. In some cases we need hyperedges of size $c+2$. Figure 3.3 shows a double circuit of degree 6 of $\mathcal{M}_{3,5}\left(\binom{V}{2}\right)$ in which the MDCP does not hold without having the hyperedge of size 3 on vertices u_1, u_2, and u_3. By contracting the singleton classes u_1u_2, u_1u_3, u_2u_3, we get a double circuit of degree 3 in $\mathcal{M}_{3,5}\left(\binom{V}{2}\right)/\{u_1u_2, u_1u_3, u_2u_3\}$ which proves that the DCP does not hold without the triple $u_1u_2u_3$.

After considering several examples, we can get that the key in the MDCP would be the ability of choosing a large independent set from any $E[X]$ (with $k|X| - l \geq 0$). The maximum size is clearly $k|X| - l$. The truth is that this is sufficient, the main statement of this chapter is the following:

Theorem 3.0.1. *If*

$$r_{\mathcal{M}_{k,l}(H)}(E[X]) = k|X| - l \text{ holds for each set } X \subseteq V \text{ with } k|X| - l \geq 0, \tag{3.2}$$

then $\mathcal{M}_{k,l}(H)$ has the MDCP.

Theorem 3.0.1 is an easy consequence of the properties of a polymatroid construction presented in Chapter 4. We will prove the MDCP in Chapter 4 in a more general context. As the proof of the general case is essentially the same, the proof of Theorem 3.0.1 is postponed for a while. Instead, we deal with the question of

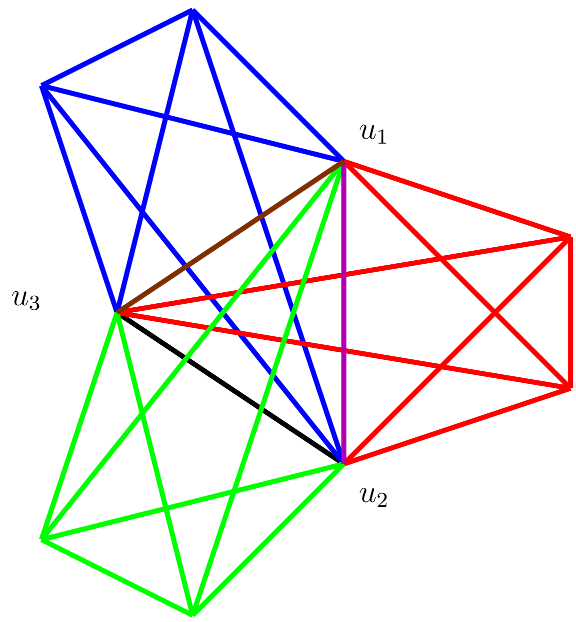

Figure 3.3: A double circuit of degree 6 in $\mathcal{M}_{3,5}\left(\binom{V}{2}\right)$

satisfying (3.2). If each set $X \subseteq V$ of size bigger than $\frac{l}{k}$ is in E with multiplicity $k|X| - l$, then (3.2) holds. A necessary and sufficient condition is stated in the following theorem, which can be proved by simple computation.

Theorem 3.0.2 ([42])**.** *Let $l = ck + d$ where c, d are integers with $0 \leq d < k$. Then, (3.2) holds if and only if E contains*

(3.3i) all the subsets of V of size $c+1$ with multiplicity at least $k-d$, and

(3.3ii) all the subsets of V of size $c+2$ with multiplicity at least $cd+d-ck$.

(Note that $cd + d - ck$ can be negative.) In other words, (3.3i) and (3.3ii) together imply that $\mathcal{F}_{E[X]}$ is composed by one single set $E[X]$ if $|X| \geq c+1$.

Proof. For the necessity, if $|X| = c+1$, then $k|X| - l = k-d$, therefore X must be a hyperedge of H with multiplicity at least $k-d$. If $|X| = c+2$, then $k|X| - l = 2k-d$. If $E[X]$ has only hyperedges of size at most $c+1$, then $E[X] \subseteq \bigcup_{v \in X} E[X - \{v\}]$, and $r_{\mathcal{M}_{k,l}}(E[X]) \leq \sum_{v \in X} k|X - \{v\}| - l = k|X| - l - (cd+d-ck)$. Therefore, each set of size $c+2$ must be in H with multiplicity at least $cd+d-ck$ if $cd+d-ck > 0$.

Now we prove the other direction by induction on $|X|$. If $|X| \le c$, then we are done. If $|X| = c + 1$, then $\mathcal{F}_{E[X]}$ is either composed by some sets $E[X - \{v\}]$ for $v \in X$ and by some singletons $\{e\}$ where $|e| = c + 1$, or $\mathcal{F}_{E[X]} = \{E[X]\}$. As there are at least $k - d = k(c + 1) - l$ hyperedges in E of size $c + 1$, we must have $\mathcal{F}_{E[X]} = \{E[X]\}$ and $b(E[X]) = k|X| - l$.

Next, let $|X| = c+2$. Again, $\mathcal{F}_{E[X]}$ is either composed by some sets $E[X - \{v\}]$ for $v \in X$ and by some singletons $\{e\}$ where $|e| = c+2$, or $\mathcal{F}_{E[X]} = \{E[X]\}$. Simple computation shows that $k|X| - l \le |\{e \in E[X] : |e| = c + 2\}| + \sum_{v \in X} k|X - \{v\}| - l$, thus the we finally have $\mathcal{F}_{E[X]} = \{E[X]\}$.

Last, let $|X| \ge c + 3$ and let $v \in X$. We know by induction, that $\mathcal{F}_{E[X-\{v\}]} = \{E[X - \{v\}]\}$ and $\mathcal{F}_{E[X']} = \{E[X']\}$ for any $X' \subseteq X$ with $v \in X'$ and $|X'| = c+2$. As $\mathcal{F}_{E[X-\{v\}]}$ and $\mathcal{F}_{E[X']}$ both refine $\mathcal{F}_{E[X]}$, and $E[X - \{v\}] \cap E[X'] \neq \emptyset$, we must have $\mathcal{F}_{E[X]} = \{E[X]\}$. $\square$

Hence, if H satisfies (3.3i) and (3.3ii), then $\mathcal{M}_{k,l}(H)$ has the MDCP.

We do not know whether (k, l)-matroids with (3.2) are pseudomodular, unless $l = 0$ or $k = l = 1$. However, we show that $\mathcal{M}_{2,3}(\binom{V}{2})$ does not have the weak series reduction property even if it satisfies (3.2). Let $V = \{x, y, u, v, z\}$, $E = \binom{V}{2}$, and let us consider the 2-dimensional rigidity matroid $\mathcal{M}_{2,3}$ on 5 vertices. Let $S = \{xy, xu, xv\}$ and $U = S \cup \binom{\{y,u,v,z\}}{2}$. Then, S is a circuit of $\mathcal{M}_{2,3}/(U - S)$ and $\mathrm{sp}_{\mathcal{M}_{2,3}}(U - S)$ is connected. Setting $T_1 = \binom{\{y,u,v,z\}}{2} - \{uv\}$ and $T_2 = \binom{\{y,u,v,z\}}{2} - \{uy\}$, we can see that $S \cup T_1$ and $S \cup T_2$ are circuits. The only $\beta_i \in E$ s.t. $\{\beta_i\} \cup T_i$ is a circuit are $\beta_1 = uv$ and $\beta_2 = uy$. Thus, there is no $\beta \in E$ which satisfies the requirement of the weak series reduction property.

Let $\mathcal{M}_{k,l}(H)$ satisfy (3.3i) and (3.3ii). Let moreover $A \subseteq 2^E$, and we ask for $\nu(f|_{2^A})$, where $f(F \cup B) = r_{\mathcal{M}}(F \cup \bigcup B)$ for $F \subseteq E$ and $B \subseteq A$. Then, we have (2.10) as in Theorem 2.6.2. In fact, this equation can be derived without the DCP of $\mathcal{M}_{k,l}(H)$ if $\mathcal{M}_{k,l}(H)$ is the sum (in the matroidal sense) of smaller (k_i, l_i)-matroids and the DCP of the smaller (k_i, l_i)-matroids is known. Say, let $k = l \ge 2$. The DCP of (hyper)graphic matroids is easier than the DCP of (k, k)-matroids in general, but a (k, k)-matroid decomposes into the sum of hypergraphic matroids. However, we have to note that if l_1/k_1 and l_2/k_2 differ, then the sum of the matroids $\mathcal{M}_{k_i,l_i}$ usually does not equal $\mathcal{M}_{k_1+k_2,l_1+l_2}$ ([21]).

For computing $\nu(f|_{2^A})$, we use a construction of Iwata, which computes the matching in a homomorphic map of a DCP matroid. As the matroid sum is the homomorphic map of the direct sum, we restrict ourselves to the homomorphic map of matroids. We restrict ourselves moreover to the special case when each member of A contains two elements of E. The general case can be derived from the restricted one. Let $\mathcal{M}$ be a matroid with ground set E with a partition $E_1, E_2, \ldots, E_m$, $T = \{t_1, t_2, \ldots t_m\}$ and the members of E_i are mapped to t_i. It is not fully conform with our polymatroidal notation, but let $\varphi(\mathcal{M})$ be the homomorphic map and let $A \subseteq 2^T$. Let $\mathcal{N}_T$ and $\mathcal{N}_{T'}$ be free matroids with ground sets T and T', where T' is a duplication of T. Let $\mathcal{N}$ be the direct sum of $\mathcal{M}$, $\mathcal{N}'_T$, and $\mathcal{N}_T$, and $A' = \{et'_i : e \in E_i, t_i \in T\} \cup \{t_i t'_i : i \in \{1, 2 \ldots, m\}\} \cup A$. Then, it is not hard to see that $\nu(r_{\varphi(\mathcal{M})}|_{2^A}) + |T| = \nu(r_{\mathcal{N}}|_{2^{A'}})$ (where the rank functions extend to the set of pairs of elements as a 2-polymatroid function). If $\mathcal{M}$ has the DCP (then so has the matroid obtained by adding free matroids), the last term has a good characterization.

Let us consider now some applications, for more on the parity problem of (k, l)-matroids, see [42].

3.1 Berge-Tutte formula and transversal matroids

As we have seen in Section 2.5, the matroid matching problem of $(1, 0)$-matroids includes the matching problem of graphs. One of the usual interpretations of transversal matroids is that we have a hypergraph $H = (V, E)$ and $F \subseteq E$ is independent if and only if $|F[X]| \leq |X|$ holds for every $X \subseteq V$, i.e. we are talking about $\mathcal{M}_{1,0}(H)$.

We have to note however, that the transversal matroid matching problem can be solved in an easier way. Tong, Lawler and Vazirani [55] showed that even the weighted case of matchings of gammoids can be reduced to the weighted matching problem of graphs. For more on gammoids, see [51].

3.2 Hypergraphic matroid

A more involved special case is the matching problem of graphic and hypergraphic matroids. We have seen that pseudomodularity and also Theorem 3.0.2 implies that $\mathcal{M}_{1,1}(H)$ has the MDCP if $\binom{V}{2} \subseteq E$. A well-known application of this is the maximum triangle cacti problem of 3-uniform hypergraphs [33], and the bit more general graphic matroid matching theorem of Lovász:

Theorem 3.2.1 (Lovász, [33]). *Let $E = \binom{V}{2}$, and $A \subseteq \binom{E}{2}$. Then,*

$$\nu_{\mathcal{M}_{1,1}}(A) = \min\left(|V| - |\mathcal{P}| + \sum_{j=1}^{t} \left\lfloor \frac{r_{\mathcal{N}}(A_j)}{2} \right\rfloor\right),$$

where the minimum is taken for all partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ of V and for all partitions $A_1, A_2, \ldots, A_t$ of A and $\mathcal{N}$ is the cycle matroid of the graph obtained from $(V, \bigcup A_j)$ by contracting the members of $\mathcal{P}$.

If $\bigcup A$ contains also hyperedges of size bigger than two, then the min-max relation cannot be rewritten in such a special form. In this case, the contraction cannot be described by a partition of V. To see this, let $e_0, e_1, \ldots, e_m$, $m \geq 3$ be pairwise vertex-disjoint hyperedges of size three and $A = \{\{e_0, e_i\} : 1 \leq i \leq m\}$. Then the only possibility of obtaining equality in the min-max formula is $Z = \{e_0\}$ and $A_j = \{\{e_0, e_j\}\}$, $1 \leq j \leq t = m$.

3.3 2-dimensional generic rigidity

Let $k = 2$ and $l = 3$. If $\bigcup A$ contains only edges, then this is the "smallest" case when Theorem 3.0.2 gives a new result. Just as above, (3.2) is satisfied if $\binom{V}{2} \subseteq E$. If E contains only edges, then it is known that the independent sets of $\mathcal{M}_{2,3}$ of rank $2|V| - 3$ are exactly the 2-dimensional minimally generically rigid graphs with vertex-set V (see Laman, [29]). Let $G = (V, E')$ be a 2-dimensional generically rigid graph and let A be a set of (not necessarily disjoint) pairs from E'. Then, we could ask for the existence of a minimally generically rigid subgraph of G composed only by edge-pairs of A. What a pity that $2|V| - 3$ is always odd. However, the maximum number of edge-pairs from A which are contained in a minimally generically rigid subgraph of G can be computed, this is $\nu_{\mathcal{M}_{2,3}}(A)$, which has a good characterization by Theorem 3.0.2. If $G = (V, E')$ is not a generically rigid graph but $(V, E' \cup \bigcup A)$ is generically rigid, where $A \subseteq \binom{\binom{V}{2}}{2}$, then $\varrho_{\mathcal{M}_{2,3}/E'}(A)$ is the minimum cardinality of a set $B \subseteq A$ s.t. $(V, E' \cup \bigcup B)$ is generically rigid. This problem has a good characterization by Theorem 3.0.2 and Theorem 2.1.2.

3.4 A forest augmentation problem

The problems discussed here were proposed by Zsolt Fekete (personal communication). Let $G = (V, E')$ be an undirected graph, let $1 \le k \le l \le 2k - 1$. Let moreover an other edge-set E'' on V and a set of packets $A \subseteq 2^{E''}$ be given. We ask for the minimum cardinality set $B \subseteq A$ s.t. $r_{\mathcal{M}_{k,l}}(V, E' \cup \bigcup B) = k|V| - l$. Clearly, if A is composed by singletons, then this is a minimum cardinality spanning subset problem in a matroid.

Frank observed (personal communication) that if each packet is composed by p parallel edges, $p = k = l$ and k is part of the input, then the problem is NP-hard. The graph on 2 vertices obtained from G after consecutively contracting $|V| - 2$ pairs of vertices contains k edge-disjoint spanning trees, if and only if G has a cut of size at least k. Hence, the maximum cut problem is included.

If $p = 2$ and $k \ge 1$, $l \ge 0$ are arbitrary integers, then we just have to compute $\varrho_{\mathcal{M}_{k,l}/E'}(A)$. This contains the problem of adding a minimum number of capacity 2 edges (from a prescribed set) to G so that the resulting graph has k edge disjoint spanning trees ($k = l$). Again, by Theorem 3.0.2 and Theorem 2.1.2, a combinatorial characterization is achieved.

An alternative way of interpreting the special case $k = l = 2$ is that we are given the graph G, a set of pairs of vertices $P \subseteq \binom{V}{2}$, and we want to choose a minimum set $P' \subseteq P$, s.t. the graph obtained from G by contracting the pairs of P' has 2 edge-disjoint spanning trees. As $G/\{u, v\}$ has 2 edge-disjoint spanning trees if and only if $G \cup \{uv^{(1)}, uv^{(2)}\}$ has any, the equivalence is clear (where $uv^{(1)}$ and $uv^{(2)}$ are parallel edges on u and v).

3.5 3-dimensional generic rigidity

The origin of examining the parity of (k, l)-matroids was suggested by Jackson and Jordán [24], which was in fact one of the starting points of this thesis. As we do not know a combinatorial characterization for the rank of the 3-dimensional generic rigidity matroid, they gave a large subset of the independence sets of the 3-dimensional generic rigidity matroid by proving that if we are given a simple graph and each vertex-set X with $|X| \ge 2$ spans at most $\frac{5|X|-7}{2}$ edges, then the edge-set of the graph is independent. This is a rather large portion of graphs with independent edge-sets, as in graphs with independent edge-sets each vertex-set X with $|X| \geq 3$ spans at most $3|X| - 6$ edges. Again, the maximum number of such set of edges in a given graph has a good characterization by Theorem 3.0.2.

Chapter 4

Solid intersecting submodular functions

Intersecting submodular functions define polymatroids in a natural way, as in (1.3xiii). This was the case for (k,l)-matroids, see [9; 11; 51] for more on the topic. Now we introduce the new abstract class of *solid* intersecting submodular functions and the polymatroids defined by them. These polymatroids will have the MDCP and the DCP, and we will get all the (k,l)-matroids having property (3.2) as a special case. The polymatroid arising in the 2-polymatroid parity formulation of Mader's vertex-disjoint $\mathcal{A}$-paths problem will be a special case, a proof of Mader's theorem based on this is presented in Chapter 5. Some parity constrained connectivity orientation problems are discussed in Chapter 6, most of the results are based on the observation that the polymatroids defined by the out-degree vectors of the good orientations can be embedded into solid polymatroids.

Let us start with an intersecting submodular function as in (1.3xiii):

(4.1i) Let S be a finite ground-set, let $\emptyset \in \mathcal{L} \subseteq 2^S$ be a family which is closed under taking intersections, and $\bigcup \mathcal{L} = S$. Let $b : \mathcal{L} \to \mathbb{Z}_+$, $b(\emptyset) = 0$ be a function having the following *intersecting submodular* property. Let us suppose that, if $U_1, U_2 \in \mathcal{L}$ with $U_1 \cap U_2 \neq \emptyset$, then there exists a member of $\mathcal{L}$ denoted by $U_1 \vee U_2$ s.t. $U_1 \cup U_2 \subseteq U_1 \vee U_2$, and

$$b(U_1) + b(U_2) \geq b(U_1 \cap U_2) + b(U_1 \vee U_2).$$

Then, $\widehat{b} : 2^S \to \mathbb{Z}_+$,

$$\widehat{b}(U) = \min_{\mathcal{F} \subseteq \mathcal{L} - \{\emptyset\},\ U \subseteq \bigcup \mathcal{F}} \sum_{U_i \in \mathcal{F}} b(U_i)$$

is a polymatroid function, and we define $\mathcal{F}_U$ for $U \subseteq S$ as in (1.3xiii). Now, we present the properties which together imply that the prematroids of $\widehat{b}$ have the MDCP.

(4.1ii) Let us suppose that if $U \in \mathcal{L} - \{\emptyset\}$, then $|\mathcal{F}_U| = 1$.

(4.1iii) Let $U_1, U_2, U_3 \in \mathcal{L}$ s.t. $b(U_{i,j}) > 0$ for every $U_{i,j} \in \mathcal{L}$ with $U_i \cap U_j \subseteq U_{i,j}$, $1 \leq i < j \leq 3$. Then, we suppose the existence of a member of $\mathcal{L}$ denoted by $\sqcup(U_1, U_2, U_3)$ s.t. $U_1 \cup U_2 \cup U_3 \subseteq \sqcup(U_1, U_2, U_3)$, and

$$\sum_{1 \leq i < j \leq 3} b(U_i \cap U_j) + b(\sqcup(U_1, U_2, U_3)) \leq \sum_{i=1}^{3} b(U_i) + b(U_1 \cap U_2 \cap U_3). \tag{4.2}$$

The quintuplet $(S, \mathcal{L}, b, \vee, \sqcup)$ is said to be *solid* if it satisfies (4.1i-4.1iii). We use the word *solid* also for the polymatroid function $\widehat{b}$ defined by a solid quintuplet. Solid polymatroid functions behave well from the viewpoint of parity. The most important properties of them are the following:

Theorem 4.0.1. *The set of solid polymatroid functions is closed under taking contractions.*

Theorem 4.0.2. *Any prematroid $\mathcal{M}_{\widehat{b}}$ of a solid polymatroid function $\widehat{b}$ is solid again.*

Theorem 4.0.3. *Solid polymatroid functions do have the MDCP.*

Therefore, Theorem 4.0.1 and 4.0.3 together imply that solid polymatroids have the DCP, and Theorem 4.0.2 and 4.0.3 proves the same for their prematroids. Hence, if $\widehat{b}$ is solid and $A \subseteq S$, then

$$\nu(\widehat{b}|_{2^A}) = \min \left(\widehat{b}(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(\widehat{b}/Z)(A_j)}{2} \right\rfloor \right),$$

where the minimum is taken for every $Z \subseteq S$ and for every partition $A_1, A_2, \ldots, A_t$ of A. It is also clear from the definition of $\widehat{b}$ that the A_j's can be decomposed into members of $\mathcal{L}_Z - \{\emptyset\}$ and we get the following:

Corollary 4.0.4. *If $\widehat{b}$ is a solid polymatroid, $A \subseteq S$, then*

$$\nu(\widehat{b}|_{2^A}) = \min\left(\widehat{b}(Z) + \sum_{j=1}^{t} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor\right),$$

where the minimum is taken for every $Z \subseteq S$ and for every family of sets $U_1, U_2, \ldots, U_t \in \mathfrak{L}_Z - \{\emptyset\}$ s.t. $A \subseteq \bigcup_{j=1}^{t} U_j$.

For better comprehension, let us recall the example of (k,l)-matroids having property (3.2). Consider the definition of $\mathfrak{L}$, b, and $\vee$ as in (3.1). Then, (3.2) means exactly (4.1ii). Moreover, if $X_1, X_2, X_3 \subseteq V$ are s.t. $k|X_i \cap X_j| - l \geq 0$ for every $1 \leq i < j \leq 3$, then

$$\sum_{1 \leq i < j \leq 3} k|X_i \cap X_j| - l + k|X_1 \cup X_2 \cup X_3| - l = \sum_{i=1}^{3} k|X_i| - l + k|X_1 \cap X_2 \cap X_3| - l,$$

hence

$$\sum_{1 \leq i < j \leq 3} k|X_i \cap X_j| - l + k|X_1 \cup X_2 \cup X_3| - l \leq \sum_{i=1}^{3} k|X_i| - l + \max\{0, k|X_1 \cap X_2 \cap X_3| - l\}.$$

Let therefore $\sqcup(E[X_1], E[X_2], E[X_3]) = E[X_1 \cup X_2 \cup X_3]$. It is easy to see that the definition of b and (4.1ii) implies (4.1iii), proving finally Theorem 3.0.1.

It is worth to see (k,l)-matroids in a bit more different, polymatroidal way. Before, we have to observe that we can assign any value to singletons which are not members $\mathfrak{L}$, and this operation preserves (4.1i) and (4.1iii):

Proposition 4.0.5. *Let $(S, \mathfrak{L}, b, \vee, \sqcup)$ satisfy (4.1i), (4.1iii), and let $s \in S$. If $\{s\} \notin \mathfrak{L}$, then let $\mathfrak{L}' = \mathfrak{L} \cup \{s\}$, $b' : \mathfrak{L}' \to \mathbb{Z}_+$, $b'|_{\mathfrak{L}} = b$. Then, after defining $b'(\{s\})$ arbitrarily, there are operators $\vee'$ and $\sqcup'$ s.t. $(S, \mathfrak{L}', b', \vee', \sqcup')$ satisfies (4.1i) and (4.1iii).*

Coming back to (k,l)-matroids, let us simply assume that $H = (V, E)$ is a hypergraph (with hyperedges of size at least $\frac{l}{k}$), and c and d are as in Theorem 3.0.2. Let $\mathfrak{L} = \{\emptyset\} \cup \{E[X] : X \subseteq V\}$, and we define $b : \mathfrak{L} \to \mathbb{Z}_+$ by $b(\emptyset) = 0$, and $b(F) = \min\{k|X| - l : X \subseteq V, F = E[X]\}$ if $F \neq \emptyset$. It is not hard to see that if E contains all subsets of V of size $c+1$ and contains moreover all subsets of size $c+2$ if $cd + d - ck > 0$, then $b(E[X]) = k|X| - l$ for $|X| \geq \frac{l}{k}$, (4.1i), (4.1ii), and (4.1iii) all hold, and finally, $\widehat{b}$ is solid. The prematroids of $\widehat{b}$ are almost the matroids what we called (k,l)-matroids earlier. However, in a prematroid of $\widehat{b}$ the multiplicities of the sets of size $c+2$ can be $k(c+2)-l$ while in a (k,l)-matroid having the MDCP, $cd+d-ck$ (which is always smaller) is enough. Moreover, we are free in prescribing the multiplicity of the set X of size bigger than $c+2$ in $\mathcal{M}_{k,l}$, while in $\mathcal{M}_{\widehat{b}}$ this multiplicity is at least $k|X|-l$. Hence, we have to modify b by introducing upper bounds on singletons which are not of form $E[X]$, we have to maintain only $b(\{e\}) \geq cd+d-ck$ if $|e|=c+2$. After adding these upper bounds, the prematroids of $\widehat{b}$ are exactly the (k,l)-matroids satisfying (3.3i) and (3.3ii). Let us call the polymatroids defined this way (k,l)-*polymatroids*.

At some point of this chapter the reader might ask as well for the explanation of the name "solid". On one hand, this is just for referencing. By a more justifiable explanation it refers to inequality (4.2) which can be interpreted that the function b does not grow too quickly. Nevertheless, in practice, (4.2) is used only to prove that the intersection of the spans of some circuits is large enough, which leads us to the MDCP.

The principal observation in the background of considering inequality (4.2) is that if $H=(V,E)$ is a hypergraph, then the same inequality holds for the supermodular function $X \mapsto |E[X]|$ with the opposite inequality sign. That is, if $X_1, X_2, X_3 \subseteq V$, then

$$\sum_{1\leq i<j\leq 3} |E[X_i \cap X_j]| + |E[X_1 \cup X_2 \cup X_3]| \geq \sum_{i=1}^{3} |E[X_i]| + |E[X_1 \cap X_2 \cap X_3]|.$$

Based on this, we can introduce other solid polymatroids. Let $S=V$, $\alpha : V \to \mathbb{Z}_+$, $l \in \mathbb{Z}_+$, let $H=(V,E)$ be a hypergraph, and let $b(X)=\alpha(X)-|E[X]|-l$ if $\emptyset \neq X \subseteq V$. (In fact, we can consider multihypergraphs, and we assume for sake of simplicity, that $|e|\geq 1$ for every $e \in E$.) We always assume moreover that α, H, and l are chosen so that $b \geq 0$. It is not hard to see that if $l=0$, then $\widehat{b}$ is the homomorphic map of a gammoid, moreover $\widehat{b}$ is solid. If $l>0$, then (4.1ii) is not necessarily satisfied. Hence, we must modify the construction by setting $S=V\cup\{uv : u,v \in V,\ u \neq v\}$, $\mathcal{L}=\{S[X] : X \subseteq V\}$, and

$$b(U) = \begin{cases} \alpha(X)-|E[X]|-l, & \text{if } U=S[X] \text{ for some } \emptyset \neq X \subseteq V, \\ 0, & \text{if } U=\emptyset, \end{cases}$$

where $S[X] = X \cup \{uv : u, v \in X,\ u \neq v\}$ for $X \subseteq V$. Then, $\widehat{b}$ is solid for every $l \geq 0$. Though $\widehat{b}$ is isomorphic to a contraction of a $(1,l)$-polymatroid, it was worth to define $\widehat{b}$ directly and see what sort of principles do make $\widehat{b}$ a good polymatroid. This construction will have an important role in Subsection 6.2.1 where we examine parity constrained rooted connected orientations.

If $(S, \mathcal{L}, b, \vee, \sqcup)$ is solid and $A \subseteq S$, then we can ask for the maximum vector x of $\mathcal{P}(\widehat{b}|_{2^A})$ having $c \leq x \leq d$ and $x \equiv \chi_T$ for some vectors $c, d : A \to \mathbb{Z}_+$ and $T \subseteq A$. Clearly, we may assume $c \equiv \chi_T$ (otherwise $c(a)$ is increased by 1 if $c(a) \not\equiv \chi_T(a)$). The problem can be solved by appropriately modifying $(S, \mathcal{L}, b, \vee, \sqcup)$. For each $a \in A$ we introduce a new member a', and let $U' = (U - A) \cup \{a, a' : a \in U \cap A\}$ for $U \subseteq S$. $\mathcal{L}' = \{\{a'\} : a \in A\} \cup \{U' : U \in \mathcal{L}\}$, $b'(\{a'\}) = d(a)$ if $a \in A$ and $b'(U') = b(U)$ if $U \in \mathcal{L}$. Let $U_1' \vee' U_2' = (U_1 \vee U_2)'$ and $\sqcup'(U_1', U_2', U_3') = (\sqcup(U_1, U_2, U_3)')'$ if $U_i \in \mathcal{L}$, while $\vee'$ and $\sqcup'$ is simply the union of the operands if one of the operands is a singleton of A'. If $c' \notin \mathcal{P}(\widehat{b'})|_{2^{A'}}$ for the vector $c' : A' \to \mathbb{Z}_+$ with $c'(a') = c(a)$, then the aimed x does not exist. Otherwise $\mathcal{P}(\widehat{b'}/c')$ is solid, and y is a maximum matching of $(\widehat{b'}/c')|_{2^{A'}}$ if and only if x with $x(a) = y(a') + c(a)$ is an answer for the original question.

Closing this chapter we proceed with the proof of Theorem 4.0.1, 4.0.2, and 4.0.3. The first two of these proofs has a technical nature. We are dealing with double circuits only in 4.0.3.

Proof of Theorem 4.0.1. The only task is to show an operator $\sqcup_z$ s.t. $(S, \mathcal{L}_z, b_z, \vee_z, \sqcup_z)$ is a solid quintuplet. This establishes the proof. Let $U_1, U_2, U_3 \in \mathcal{L}_z$ with $\widehat{b}_z(U_i \cap U_j) > 0$, $1 \leq i < j \leq 3$. Then,

$$b_z(U_i \cap U_j) = b(U_i \cap U_j) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_i \cap U_j]} b(X) - z(U_i \cap U_j - \mathrm{sp}_{\widehat{b}}(z))$$

for any $1 \leq i < j \leq 3$. If $U_1 \cap U_2 \cap U_3 \neq \emptyset$, then

$$\mathcal{F}_{(U_1 \cap U_2 \cap U_3) \cup \mathrm{sp}_{\widehat{b}}(z)} = \{U_1 \cap U_2 \cap U_3\} \cup (\mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)} - \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_1 \cap U_2 \cap U_3]),$$

and

$$\mathcal{F}_{(U_1 \cap U_2 \cap U_3) \cup \mathrm{sp}_{\widehat{b}}(z)} = \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)},$$

if $U_1 \cap U_2 \cap U_3 = \emptyset$. In both cases,

$$b_z(U_1 \cap U_2 \cap U_3) = b(U_1 \cap U_2 \cap U_3) - \sum_{X \in \mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[U_1 \cap U_2 \cap U_3]} b(X) - z(U_1 \cap U_2 \cap U_3 - \mathrm{sp}_{\widehat{b}}(z)).$$

By applying (4.1iii) to b, and U_1, U_2, U_3, we have

$$\sum_{1\le i<j\le 3} b_z(U_i \cap U_j) + b(\sqcup(U_1,U_2,U_3)) - \sum_{X\in\mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[\sqcup(U_1,U_2,U_3)]} b(X) - z(\sqcup(U_1,U_2,U_3) - \mathrm{sp}_{\widehat{b}}(z)) \le \sum_{i=1}^{3} b_z(U_i) + b_z(U_1\cap U_2\cap U_3).$$

By Proposition 1.4.2, there exists a set $U' \in \mathcal{L}_z$ s.t. $U' \supseteq \sqcup(U_1,U_2,U_3)$, and $b_z(U') \le b(\sqcup(U_1,U_2,U_3)) - \sum_{X\in\mathcal{F}_{\mathrm{sp}_{\widehat{b}}(z)}[\sqcup(U_1,U_2,U_3)]} b(X) - z(\sqcup(U_1,U_2,U_3) - \mathrm{sp}_{\widehat{b}}(z))$, therefore let $\sqcup_z(U_1,U_2,U_3) = U'$. □

Proof of Theorem 4.0.2. We define a solid quintuplet $(E, \mathcal{L}', b', \vee', \sqcup')$ s.t. $\widehat{b'} = r_{\mathcal{M}_{\widehat{b}}}$. Let $\mathcal{L}' = \{\emptyset\} \cup \{\{e\} : e \in E\} \cup \{\bigcup_{s\in U} E_s : U \in \mathcal{L}\}$, and

$$b'(F) = \begin{cases} b(U), & \text{if } F = \bigcup_{s\in U} E_s, U \in \mathcal{L}, |F| \ne 1 \\ \min(1, b(U)), & \text{if } F = \bigcup_{s\in U} E_s, U \in \mathcal{L}, |F| = 1 \\ 1, & \text{otherwise, if } |F| = 1, \\ 0, & \text{if } F = \emptyset. \end{cases}$$

It is not hard to see that there are operators $\vee'$ and $\sqcup'$ with (4.1i) and (4.1iii). □

Proof of Theorem 4.0.3. It is not hard to see that if c is a circuit of $\widehat{b}$ with $r_{\widehat{b}}(c) > 0$, then $\mathrm{supp}(c) \subsetneq \mathrm{sp}(c)$ and $|\mathcal{F}_{\mathrm{sp}(c)} - \mathcal{F}_\emptyset| = 1$. Let x be a NTCDC of $\widehat{b}$ with principal partition $U_1, U_2, \ldots, U_d$, and with circuits x_i as defined in (2.12). For $T \subseteq [d]$, let $C(T) = \bigcap_{t\in T} \mathrm{sp}(x_t)$ if $T \ne \emptyset$, and $C(\emptyset) = \mathrm{sp}(x)$. Then, $|\mathcal{F}_{C(T)} - \mathcal{F}_\emptyset| \le 1$ and $C(T) \subseteq \bigcup \mathcal{F}_{C(T)}$.

Lemma 4.0.6. *If x is a NTCDC of $\widehat{b}$ with the above notations, $T \subseteq [d]$, then*

$$\bigcup \mathcal{F}_{C(T-i)\cup C(T-j)} = C(T - \{i,j\}), \tag{4.3}$$

where $i, j \in T$, $i \ne j$, and

$$\widehat{b}(C(T)) = \sum_{t\in[d]-T} x(U_t) + |T| - 2. \tag{4.4}$$

Proof. The statement is proved by induction on $|T|$. For $|T| \leq 1$, only (4.4) is to be proved, which is clear. Let $T = \{i, j\}$. As $x = x_i \vee x_j$, (4.3) follows. For (4.4), let U_i be the unique member of $\mathcal{F}_{C(i)} - \mathcal{F}_\emptyset$. As $x_i \wedge x_j \in \mathcal{P}(\widehat{b})$, we have $(x_i \wedge x_j)(S) \leq \widehat{b}(C(\{i,j\})) \leq b(U_i \cap U_j) \leq b(U_i) + b(U_j) - b(U_i \vee U_j) \leq x_i(S) - 1 + x_j(S) - 1 + x(S) - 2$.

So let us assume $|T| \geq 3$ and $T = [|T|]$ for sake of simplicity. First, (4.3) is proved. It can be seen immediately that

$$C(T - i) \cup C(T - j) \subseteq C(T - \{i, j\}). \tag{4.5}$$

Next we apply submodularity to $C(T - i) \cup C(T - j)$ and $C(i)$. As

$$\bigcup \mathcal{F}_{C(T-i) \cup C(T-j) \cup C(i)} = C(\emptyset),$$

and

$$(C(T - i) \cup C(T - j)) \cap C(i) \supseteq C(T - j),$$

we get

$$\begin{aligned} \widehat{b}(C(T - i) \cup C(T - j)) + \widehat{b}(C(i)) \geq \\ \widehat{b}(C(T - j)) + \widehat{b}(C(\emptyset)) = \widehat{b}(C(T - \{i, j\})) + \widehat{b}(C(i)), \end{aligned} \tag{4.6}$$

where the last equality is obtained by using the induction hypothesis, (4.4). As $C(T - \{i, j\})$ is a flat, (4.5) and (4.6) together gives $\bigcup \mathcal{F}_{C(T-i) \cup C(T-j)} = C(T - \{i, j\})$, thus (4.3) is proved.

For (4.4),

$$\widehat{b}(C(T)) \leq \sum_{t \in [d] - T} x(U_t) + |T| - 2$$

holds by Claim 2.6.3. For the reverse inequality, we apply (4.1iii) to $\{U_1\} = \mathcal{F}_{C(T-\{2,3\})} - \mathcal{F}_\emptyset$, $\{U_2\} = \mathcal{F}_{C(T-\{1,3\})} - \mathcal{F}_\emptyset$, and $\{U_3\} = \mathcal{F}_{C(T-\{1,2\})} - \mathcal{F}_\emptyset$. Then,

$$\widehat{b}(C(T)) = b(\bigcap_{i \in [3]} U_i) \geq \sum_{\{i,j\} \in \binom{[3]}{2}} b(U_i \cap U_j) + b(\sqcup(U_1, U_2, U_3)) - \sum_{i \in [3]} b(U_i).$$

As $\{U_1 \cap U_2\} = \mathcal{F}_{C(T-\{3\})} - \mathcal{F}_\emptyset$, we have $b(U_1 \cap U_2) = \widehat{b}(U_1 \cap U_2)$, and similarly, $b(U_1 \cap U_3) = \widehat{b}(U_1 \cap U_3)$ and $b(U_2 \cap U_3) = \widehat{b}(U_2 \cap U_3)$. As $\bigcup \mathcal{F}_{\sqcup(U_1,U_2,U_3)} \supseteq C(T - [3])$, we have $b(\sqcup(U_1, U_2, U_3)) \geq \widehat{b}(C(T - [3]))$. Again, we know all the quantities on the right hand side, which together yield

$$\widehat{b}(C(T)) \geq \sum_{i \in [d] - T} x(U_i) + |T| - 2.$$

□

□

Chapter 5

Mader's $\mathcal{A}$-paths

In this chapter we show how to prove Mader's vertex-disjoint $\mathcal{A}$-path theorem (Mader, [41]) with the help of polymatroid matching of a solid polymatroid. First, let $G = (V, E)$ be an undirected graph and let $\mathcal{A} = \{T_1, T_2, \ldots, T_k\}$ be a set of pairwise disjoint subsets of V, the members of $\mathcal{A}$ are called *terminal sets*. A path of G is said to be an $\mathcal{A}$-path if its end-vertices belong to different parts of $\mathcal{A}$, and these are the only vertices on the path which belong to $\bigcup \mathcal{A}$. Mader gave a combinatorial min-max relation for the maximum number of vertex-disjoint $\mathcal{A}$-paths. This number is denoted by $\mu(G, \mathcal{A})$.

As a surprising application of his theory, Lovász [33] has shown that Mader's problem can be formulated as polymatroid matching. For this, he defined the 2-polymatroid function $\widehat{p}$ with $\widehat{p}(F) = \min_{U_1,U_2,\ldots,U_t \subseteq V,\ F \subseteq \bigcup_{i=1}^{t} E[U_i]} p(U_i)$ for $F \subseteq E$, where $p : 2^V \to \mathbb{Z}_+$,

$$p(X) = \begin{cases} 2|X| - 2, & \text{if } \emptyset \neq X \subseteq V - \bigcup \mathcal{A}, \\ |X| - 1 + |X - \bigcup \mathcal{A}|, & \text{if } X \subseteq V \text{ meets exactly one of } T_1, T_2, \ldots, T_k, \\ |X| + |X - \bigcup \mathcal{A}|, & \text{if } X \subseteq V \text{ meets at least two of } T_1, T_2, \ldots, T_k. \end{cases}$$

For sake of simplicity, let us assume that G is connected and we have at least two non-empty terminal sets. Simple computation shows that $\mu(G, \mathcal{A}) = \nu(\widehat{p}) - |V| + |\bigcup \mathcal{A}|$ (Lovász [33]). Lovász characterized the even NTCDCs of the 2-polymatroid $\widehat{p}$ and gave a proof of Mader's min-max relation by following an inductive approach. That proof reduces the problem into smaller ones with the help of Theorem 2.3.1.

Later, Schrijver [52] gave an explicit linear representation of $\widehat{p}$, therefore, this approach gives a polynomial algorithm to compute $\mu(G,\mathcal{A})$.

We will show that $\widehat{p}$ can be embedded into a small solid polymatroid. Contrary to the linear approach, the advantage of this one is the ability of deriving Mader's combinatorial characterization to $\mu(G,\mathcal{A}$ from the characterization of $\nu(\widehat{p})$. Moreover, if there would be a polymatroid parity algorithm for DCP or solid polymatroids, then that could be applied to this embedding. The resulting algorithm would have a combinatorial manner, without dealing with linear elements but with simple combinatorial objects. It is a sore spot that are not aware of such an algorithm.

Let us consider now the key question of embedding $\widehat{p}$ into a solid polymatroid. For each $0 \le h \le k$, let us put a singleton to v, denoted by v_h. Let $S_0 = \{v_0 : v \in V\} \cup \{uv : u, v \in V,\ u \ne v\}$, $S_i = S_0 \cup \{v_i : v \in V\}$ for $i \in [k]$, and $S = S_\infty = S_0 \cup \{v_i : v \in V,\ i \in [k]\}$. Hence, $|S| = \binom{|V|}{2} + (k+1)|V|$. For $X \subseteq V$ and $h \in \{0, 1, 2, \dots, k, \infty\}$, we use the notation $S_h[X] = \{e \subseteq X : e \in S_h\}$. Let $\mathcal{L} = \{S_0[X], S_i[X], S_\infty[X] : X \subseteq V,\ i \in [k]\}$, and let $b : \mathcal{L} \to \mathbb{Z}_+$ be defined by

$$b(F) = \begin{cases} 2|X| - 2, & \text{if } F = S_0[X] \text{ for some } \emptyset \ne X \subseteq V, \\ 2|X| - 1, & \text{if } F = S_i[X] \text{ for some } \emptyset \ne X \subseteq V \text{ and } i \in [k], \\ 2|X|, & \text{if } F = S_\infty[X] \text{ for some } X \subseteq V. \end{cases}$$

Simple computation shows that $\widehat{b}$ is a solid polymatroid. Moreover, for $Z = \bigcup_{i\in[k]}\{v_i : v \in T_i\}$, we have $(\widehat{b}/Z)|_{2^E} = \widehat{p}$. Hence, the remaining task is to derive Mader's theorem from the characterization of $\nu((\widehat{b}/Z)|_{2^E})$.

Theorem 5.0.7 (Mader [41]). *The maximum number of vertex-disjoint $\mathcal{A}$-paths of the graph $G = (V, E)$ is*

$$\min\left(|R| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{|C \cap \bigcup_{i\in[k]} X_i|}{2} \right\rfloor\right),$$

where $R, X_1, \dots, X_k$ are pairwise disjoint subsets of V, $T_i \subseteq R \cup X_i$ for each $i \in [k]$ and $\mathcal{C}$ denotes the family of vertex-sets of the components of $G - R - \bigcup_{i=1}^k E[X_i]$.

Proof. We derive the non-trivial max $\ge$ min part. By Corollary 4.0.4, we have

$$\nu((\widehat{b}/Z)|_{2^E}) = \min\left((\widehat{b}/Z)(W) + \sum_{j=1}^{t} \left\lfloor \frac{b_W(F_j)}{2} \right\rfloor\right), \tag{5.1}$$

where the minimum is taken for all $W \subseteq S$ with $Z \subseteq W$ and for every family $F_1, F_2, \ldots, F_t \in \mathcal{L}_W - \{\emptyset\}$ with $E \subseteq \bigcup_{j=1}^{t} F_j$.

Let $\mathcal{F}_F = \bigcup_{h \in \{0,1,\ldots,k,\infty\}} \mathcal{F}_{h,F}$, where the members of $\mathcal{F}_{h,F}$ are of form $S_h[X]$ resp. For technical reason, let $\mathcal{X}_{h,F} = \{\emptyset \neq X \subseteq V : S_h[X] \in \mathcal{F}_{h,F}\}$. It can be observed that $\mathcal{X} = \bigcup_{h \in \{0,1,2,\ldots,k,\infty\}} \mathcal{X}_{h,F}$ is formed by pairwise disjoint sets, and $|\mathcal{X}_{\infty,F}| \leq 1$. Let us choose W so that $|\{X \in \mathcal{X}_{h,W} : |X| \geq 2, 0 \leq h \leq k\}|$ is as small as possible.

We give new indexing of the F_j's, as $F_{h,l}$, $h \in \{0,1,2,\ldots,k,\infty\}$, $l \in [t_i]$, where $F_{h,l} = S_h[X_{h,l}]$. Let us choose the $F_{h,l}$'s so that $\sum_{h=0}^{k} t_h + t_\infty$ is as small as possible.

Claim 5.0.8. *For each $F \in \mathcal{F}_W$ not of form $S_i[\{v\}]$, $v \in T_i$, $i \in [k]$, there exist $F_{h',l'} \neq F_{h'',l''}$ s.t. $F \subsetneq F_{h',l'}$ and $F \subsetneq F_{h'',l''}$.*

Proof. If there is no $F_{h',l'}$ with $F \subseteq F_{h',l'}$, then we could replace $\mathcal{F}_W$ by removing F, adding $S_i[\{v\}]$ for $v \in T_i,\ v_i \in F$, and $S_0[\{v\}]$ for $v \notin \bigcup T_i,\ v_0 \in F$, and by increasing t by setting $F_{t+1} = F$. If there is only one such set, then we simply replace $\mathcal{F}_W$ by removing F, adding $S_i[\{v\}]$ for $v \in T_i,\ v_i \in F$, and $S_0[\{v\}]$ for $v \notin \bigcup T_i,\ v_0 \in F$. In both cases, this would contradict the choice of W. □

Claim 5.0.9. $t_0 = 0$.

Proof. As G is connected, $(V, \mathcal{X} = \{X_{h,l} : h \in \{0,1,2,\ldots,k,\infty\},\ l \in [t_h]\})$ is a connected hypergraph. If $X_0 \in \mathcal{X}_0$, then by $\bigcup \mathcal{A} \neq \emptyset$ there exist $X \in \mathcal{X} - \mathcal{X}_0$ with $X \cap X_0 \neq \emptyset$. But then, we could remove X_0 from $\mathcal{X}$, and replace X by $X \cup X_0$. This contradicts the minimality of $\sum_{h=0}^{k} t_h + t_\infty$. □

Proposition 5.0.10. *If $F_{h',l'} \cap F_{h'',l''} \neq \emptyset$, then $F_{h',l'} \cap F_{h'',l''} \in \mathcal{F}_W$. Thus, we have the following possibilities.*

(5.2i) If $F_{h',l'} \cap F_{h'',l''} \in \mathcal{F}_{0,W}$, then $h', h'' \in [k]$, and $h' \neq h''$.

(5.2ii) If $F_{h',l'} \cap F_{h'',l''} \in \mathcal{F}_{i,W}$, $i \in [k]$, then $h' = h'' = i$ or $\{h', h''\} = \{i, \infty\}$.

(5.2iii) If $F_{h',l'} \cap F_{h'',l''} \in \mathcal{F}_{\infty,W}$, then $h' = h'' = \infty$.

As we have two non-empty terminal sets, and G is connected, then $\mathcal{X}_{\infty,W} \neq \emptyset$. Let R be its unique member.

Claim 5.0.11. $|X| = 1$ *for every* $X \in \mathcal{X}_{h,W}$, $0 \leq h \leq k$. *Moreover, we can choose the dual solution so that* $b_W(F_{i,l}) \leq 1$ *for every* $i \in [k]$, $l \in [t_i]$.

Proof. If the statement does not hold, then we replace W by W' defined by

$$R' = R \cup \bigcup \{X_{i,l} \cap X_{i',l'} : i, i \in [k], i \neq i'\},$$
$$\mathcal{X}_{i,W'} = \left\{ \{v\} : v \in \bigcup_{l=1}^{t_i} \left(X_{i,l} - \bigcup_{i' \in [k]-i} X_{i',l'} \right) \right\},$$
$$\mathcal{X}_{0,W'} = \left\{ \{v\} : v \in V - R' - \bigcup \bigcup_{i \in [k]} \mathcal{X}_{i,W'} \right\}.$$

Then, for $i \in [k]$ and $l \in [t_i]$, we have

$$b_W(F_{i,l}) = 2|X_{i,l}| - 1 - \sum_{X \in \mathcal{X}_{0,W}[X_{i,l}]} (2|X| - 2) - \sum_{X \in \mathcal{X}_{i,W}[X_{i,l}]} (2|X| - 1),$$

and

$$\left\lfloor \frac{b_W(F_{i,l})}{2} \right\rfloor =$$
$$|X_{i,l} - \bigcup \mathcal{X}_{0,W}[X_{i,l}] - \bigcup \mathcal{X}_{i,W}[X_{i,l}]| + |\mathcal{X}_{0,W}[X_{i,l}]| + \left\lfloor \frac{-1 + |\mathcal{X}_{i,W}[X_{i,l}]|}{2} \right\rfloor \geq$$
$$|X_{i,l} - \bigcup \mathcal{X}_{0,W}[X_{i,l}] - \bigcup \mathcal{X}_{i,W}[X_{i,l}]| + |\mathcal{X}_{0,W}[X_{i,l}]|.$$

For the last inequality we use $\mathcal{X}_{i,W}[X_{i,l}] \neq \emptyset$. By Claim 5.0.8 and Proposition 5.0.10, every member of $\bigcup_{i=1}^{k} \bigcup_{l=1}^{t_i} \mathcal{X}_{0,W}[X_{i,l}]$ is contained by at least two different $X_{i',l'}$'s, which means $\sum_{i=1}^{k} \sum_{l=1}^{t_i} |\mathcal{X}_{0,W}[X_{i,l}]| \geq 2 \left| \bigcup_{i=1}^{k} \bigcup_{l=1}^{t_i} \mathcal{X}_{0,W}[X_{i,l}] \right|$. For $l \in [t_\infty]$, we have

$$b_W(F_{\infty,l}) = 2|X_{\infty,l} - R| - \sum_{i=1}^{k} \sum_{X \in \mathcal{X}_{i,W}[X_{\infty,l}]} (2|X| - 1),$$

and

$$\left\lfloor \frac{b_W(F_{\infty,l})}{2} \right\rfloor \geq$$

$$\sum_{i=1}^{k} \sum_{X \in \mathcal{X}_{i,W}[X_{\infty,l}]} (1 - |X|) + \left\lfloor \frac{2|X_{\infty,l} - R| - \sum_{i=1}^{k} \sum_{X \in \mathcal{X}_{i,W}[X_{\infty,l}]} |X|}{2} \right\rfloor =$$

$$\sum_{i=1}^{k} \sum_{X \in \mathcal{X}_{i,W}[X_{\infty,l}]} (1 - |X|) + \left\lfloor \frac{b_{W'}(F_{\infty,l})}{2} \right\rfloor.$$

As

$$\sum_{X \in \mathcal{X}_{0,W}} (2|X| - 2) + 2|R| + 2\left|\bigcup_{i=1}^{k} \bigcup_{l=1}^{t_i} \mathcal{X}_{0,W}[X_{i,l}]\right| = 2|R'|,$$

and

$$\sum_{X \in \mathcal{X}_{i,W}} (2|X| - 1) + \sum_{X \in \mathcal{X}_{i,W}[X_{\infty,l}]} (1 - |X|) + \left|\bigcup_{l=1}^{t_i} \left(X_{i,l} - \bigcup \mathcal{X}_{0,W}[X_{i,l}] - \bigcup \mathcal{X}_{i,W}[X_{i,l}]\right)\right| = \left|\bigcup \mathcal{X}_{i,W'}\right|,$$

we have

$$\widehat{b}(W) + \sum_{i=1}^{k} \sum_{l=1}^{t_i} \left\lfloor \frac{b_W(F_{i,l})}{2} \right\rfloor + \sum_{l=1}^{t_\infty} \left\lfloor \frac{b_W(F_{\infty,l})}{2} \right\rfloor \geq$$

$$\sum_{i=1}^{k} \left|\bigcup \mathcal{X}_{i,W'}\right| + 2\,|R'| + \sum_{l=1}^{t_\infty} \left\lfloor \frac{b_{W'}(F_{\infty,l})}{2} \right\rfloor = \widehat{b}(W') + \sum_{l=1}^{t_\infty} \left\lfloor \frac{b_{W'}(F_{\infty,l})}{2} \right\rfloor.$$

If $E \subseteq \bigcup_{l=1}^{t_\infty} \mathcal{F}_{F_{\infty,l} \cup W'}$, then we get a valid dual solution. Each $uv \in E$ which is not covered by these sets, were covered originally by some of the $\mathcal{F}_{F_{i,l}}$'s, $i \in [k]$. In this case, let us cover uv by the F, where $\{F\} = \mathcal{F}_{\{uv\} \cup W'} - \mathcal{F}_{W'}$. By the construction of W', either $u, v \in \bigcup \mathcal{X}_{i,W'}$, or $u \in \bigcup \mathcal{X}_{i,W'}$ and $v \in R'$, or $u, v \in R'$. This implies $b_{W'}(F) \leq 1$ and $\left\lfloor \frac{b_{W'}(F)}{2} \right\rfloor = 0$. Hence we have a valid dual solution. □

For $i \in [k]$, let $X_i = \bigcup \mathcal{X}_{i,W}$. By Proposition 5.0.10, if $l \neq l'$, then $X_{\infty,l} \cap X_{\infty,l'} = R$. To establish the proof, we have to compute

$$\mu(G, \mathcal{A}) = -|V| + \left|\bigcup \mathcal{A}\right| + \widehat{b_Z}(W) + \sum_{l=1}^{t_\infty} \left\lfloor \frac{b_W(F_{\infty,l})}{2} \right\rfloor.$$

First,

$$-|V| + \left|\bigcup \mathcal{A}\right| + \widehat{b_Z}(W) =$$

$$-|V|+\sum_{i=1}^{k}|T_i|+\sum_{i=1}^{k}|X_i-X_i\cap T_i|+2|R|-\sum_{i=1}^{k}|R\cap T_i| = |R|-\left|V-R-\bigcup\nolimits_{i=1}^{k} X_i\right|.$$

Next,

$$\sum_{l=1}^{t_\infty}\left\lfloor\frac{b_W(F_{\infty,l})}{2}\right\rfloor = \sum_{l=1}^{t_\infty}\left\lfloor\frac{2|X_{\infty,l}-R|-\sum_{i=1}^{k}|X_{\infty,l}\cap X_i|}{2}\right\rfloor =$$

$$\sum_{l=1}^{t_\infty}\left(|X_{\infty,l}-R-\bigcup_{i=1}^{k}X_i| + \left\lfloor\frac{\sum_{i=1}^{k}|X_{\infty,l}\cap X_i|}{2}\right\rfloor\right).$$

Clearly, $X_{\infty,l} - R$ are pairwise disjoint and they together cover $E[V-R] - \bigcup_{i=1}^{k} E[X_i]$. Setting $C_l = X_{\infty,l} - R$, the corresponding minimum is at least

$$\min\left(|R| + \sum_{l=1}^{t_\infty}\left\lfloor\frac{|C_l\cap\bigcup_{i=1}^{k}X_i|}{2}\right\rfloor\right).$$

If some C_l spans a non-connected graph in $E[V-R] - \bigcup_{i=1}^{k} E[X_i]$, then we can divide it into components which does not increase the minimum. $\square$

Chapter 6

Parity constrained connectivity orientations

Connectivity orientations of (hyper)graphs and matchings of graphs have a natural common generalization, the field of parity constrained connectivity orientations. This means that an orientation of the given hypergraph $H = (V, E)$ is to be obtained which meets some connectivity requirements and moreover has a prescribed parity of out-degree for each vertex. In what follows a hyperedge can contain a vertex multiple times, so a hyperedge $e \in E$ is identified with the multiplicity function $e : V \to \mathbb{Z}_+$, i.e. e contains v with multiplicity $e(v)$. By $E[X]$ we mean the set of hyperedges e having $\mathrm{supp}(e) \subseteq X$.

Let us assume, that $|e| \geq 1$ for every $e \in E$. By an *orientation* $\vec{H}$ of H we mean that for each hyperedge $e \in E$ a vertex $v \in e$ is chosen which is called the *head* of e (and is denoted by $\mathrm{head}(e)$) and the vertices of the hyperedge $e - \mathrm{head}(e)$ are the *tail*s (and this is denoted by $\mathrm{tail}(e)$). A hyperedge e *enter*s X if X contains the head of e (i.e. $\sum_{v \in X} \mathrm{head}(e)(v) > 0$) and does not contain all of its tails (i.e. $\mathrm{supp}(\mathrm{tail}(e)) \not\subseteq X$). Similarly, e *leave*s X if it enters $V - X$. The set of hyperedges entering and leaving X is denoted resp. by $\delta^{in}_{\vec{H}}(X)$ and $\delta^{out}_{\vec{H}}(X)$.

The connectivity requirement is prescribed by the function $p : 2^V \to \mathbb{Z}_+$. We always assume that $p(\emptyset) = p(V) = 0$ and remember that p is non-negative. An orientation $\vec{H}$ of H is said to *cover* p if $\delta^{in}_{\vec{H}}(X) \geq p(X)$ for every $X \subseteq V$. We say that the function $p : 2^V \to \mathbb{Z}_+$ is *intersecting supermodular* if

$$p(X) + p(Y) \geq p(X \cap Y) + p(X \cup Y) \tag{6.1}$$

holds for every $X, Y \subseteq V$ with $X \cap Y \neq \emptyset$, and *co-intersecting supermodular* if (6.1) holds whenever $X, Y \subseteq V$ with $X \cup Y \neq V$. Similarly, $p : 2^V \to \mathbb{Z}_+$ is *crossing supermodular* if (6.1) holds for every $X, Y \subseteq V$ with $X \cap Y \neq \emptyset$ and $X \cup Y \neq V$.

Let $g : 2^V \to \mathbb{Z}$, $g(X) = \sum_{e \in E} e(X) - |E[X]| - p(X)$. It is clear, that if $x : V \to \mathbb{Z}$ is the out-degree vector of an orientation covering p, then

$$x \geq 0, \tag{6.2}$$

$$x(X) \leq g(X), \quad \text{for every } \emptyset \neq X \subseteq V, \tag{6.3}$$

$$x(V) = g(V). \tag{6.4}$$

Moreover, the opposite direction is also true due to the following simple observation (see e.g. Frank, Király, and Király [14]).

Lemma 6.0.12. *Let $H = (V, E)$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a function s.t. $p(\emptyset) = p(V) = 0$, and $x : V \to \mathbb{Z}_+$. Then, H has an orientation covering p s.t. the out-degree of each vertex $v \in V$ is $x(v)$, if and only if (6.3) and (6.4) hold.*

Orientation problems with crossing supermodular connectivity requirements have a wide literature. If p is crossing supermodular, and $g \geq 0$, then the constraints (6.2-6.4) determine a base polyhedron. Hence, for an orientation covering p, we need an integer vector from this base polyhedron. For an orientation covering p, with even out-degrees, we need an even vector with the same properties. There are also some solvable connectivity orientation problems with requirements that are not crossing supermodular, but in those cases we lose the important structural property that the out-degree vectors of the good orientations form base polyhedra. That is why we restrict ourselves in this way. The following can be proved using basic properties of polymatroids:

Theorem 6.0.13. *Let $H = (V, E)$ be a hypergraph and let $p : 2^V \to \mathbb{Z}_+$, $p(\emptyset) = p(V) = 0$. If p is intersecting supermodular, then H has an orientation covering p, if and only if*

$$g(V) \leq \sum_{j=1}^{t} g(X_j) \tag{6.5}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V. If p is co-intersecting supermodular, then H has an orientation covering p, if and only if

$$(t-1) g(V) \leq \sum_{j=1}^{t} g(V - X_j) \tag{6.6}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V.

The characterization for the more general orientation problem when the connectivity requirement is described by a non-negative crossing supermodular function is given by Frank, Király, and Király [14]:

Theorem 6.0.14. *Let $H = (V, E)$ be a hypergraph and let $p : 2^V \to \mathbb{Z}_+$ be a crossing supermodular function with $p(\emptyset) = p(V) = 0$. Then, H has an orientation covering p, if and only if (6.5) and (6.6) holds for every partition $X_1, X_2, \ldots, X_t$ of V.*

Now we turn to the parity constrained case. Hence, let $T \subseteq V$, and we are looking for connectivity orientations having odd out-degrees exactly in the vertices of T. There are natural parity involving strengthening of the inequalities (6.5) and (6.6) which surprisingly give characterizations in some cases, however there is no general recipe. It is clear that if x is the out-degree vector of a parity constrained orientation and $X \subseteq V$, then $x(X) \leq g(X)$ if $g(X) \equiv |T \cap X|$, but the stronger inequality $x(X) \leq g(X) - 1$ holds if $g(X) \not\equiv |T \cap X|$. Hence, if there is a parity constrained orientation, then

$$g(V) \leq \sum_{j=1}^{t} g(X_j) - |\{j : g(X_j) \not\equiv |T \cap X_j|\}|, \tag{6.7}$$

and

$$(t-1)g(V) \leq \sum_{j=1}^{t} g(V - X_j) - |\{j : g(V - X_j) \not\equiv |T - X_j|\}| \tag{6.8}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V. Notice that any of (6.7) or (6.8) imply that $g(V) \equiv |T|$ (by applying to the partition $(\emptyset, V)$). There are cases when we do not know the sufficiency of these formulas, there are characterized cases where they do not seem to be sufficient, and last, there are open cases where they are not sufficient.

We have to note that if H is a graph, then by setting $p'(X) = p(V - X)$ and $T' = \{v : \chi_T(v) \not\equiv \delta(v)\}$, H has an orientation which is good for p and T if and only if H has an orientation which is good for p' and T' and vice versa. Namely, given an orientation for one of these problems, the reversed oriented graph is good for the other one. Hence the co-intersecting supermodular problem is equivalent to the intersecting supermodular case as they go to each other by this simple construction. In the hypergraph case this correspondence does not work due to the asymmetry of having one head and multiple tails in the hyperedges.

We will distinguish the cases when p is intersecting, co-intersecting, or crossing supermodular, as the difficulty of the latter problems is rather different. We also examine the difficulties occurring when we add lower and upper out-degree bounds to tractable connectivity requirements. E.g. if p is intersecting supermodular, then we can add upper bounds to the out-degrees and keeping p intersecting supermodular. When we add lower bounds, then p becomes crossing supermodular, however this operation causes smaller difficulties as compared to considering general crossing supermodular requirements.

6.1 Requirement on singletons and their complements

Let us start with the easiest cases when p can be positive only on singletons and their complements. In other words, the connectivity requirement prescribes only lower and upper bounds on the out-degrees. The parity constrained problem with this kind of connectivity requirement was considered by Frank, Sebő, and Tardos [15] for graphs. They have shown that the problem reduces to the matching problem of graphs, and gave a characterization which is similar to Tutte's theorem in the sense that we have to count the components having "bad parity" after deleting some set. It is not hard to see that the same method works for the hypergraph case. Let $0 \leq c(v) \leq d(v) \leq \sum_{e \in E} e(v) - |E[v]|$ be the lower and upper bound for the out-degree of v respectively.

A possible characterization for the hypergraph case in a polymatroidal language is essentially the following. Let g_0 be defined similarly to g with $p = 0$. Then, $\widehat{g_0}$ is solid. We have mentioned in Chapter 4 that the problem when we are looking for a vector $x \in \mathcal{P}(\widehat{g_0})$ with $c \leq x \leq d$ and $x \equiv c$ can be reduced to the parity problem of a solid polymatroid. This leads to a characterization. The obtained characterization can be reformulated in our dearly-loved form. First, the case having only upper bounds in the out-degrees:

Theorem 6.1.1. *Let $H = (V, E)$ be a hypergraph, let $p : 2^V \to \mathbb{Z}_+$, suppose that p is 0 except on singletons, and finally let $T \subseteq V$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if (6.7) holds for every partition $X_1, X_2, \ldots, X_t$ of V.*

We will see in Section 6.2 that the partition formula (6.7) gives a characterization even in the more general case when p is a non-negative intersecting supermodular function.

In the case when we have only lower bounds on the out-degrees i.e. $p(X) = 0$ if X is not the complement of a singleton, the easier copartition formula (6.8) would be a candidate. However we do not know whether it is sufficient. For graphs it is sufficient, as the problem is equivalent to the case with only upper bounds:

Theorem 6.1.2. *Let $H = (V, E)$ be a graph, let $p : 2^V \to \mathbb{Z}_+$ s.t. p can be positive only on complements of singletons, and finally let $T \subseteq V$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if (6.8) holds for every partition $X_1, X_2, \ldots, X_t$ of V.*

In the case when there are lower and upper bounds on the out-degrees we do not think that there is a really nice human-comprehensible formula containing only partitions, copartitions, and similar animals. It is noted in [15] however, that even in the graph case (6.7) and (6.8) are not sufficient. For this, let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, the lower bounds are $(0, 0, 2, 0)$, the upper bounds are $(0, 2, 2, 2)$, and all the out-degrees must be even. Then, there exists a parity constrained orientation with only the lower bounds, and there is another one with the upper bounds. Hence, (6.7) and (6.8) holds. But there is no parity constrained orientation with the lower and upper bounds.

6.2 Intersecting supermodular orientations

Nebeský [44] studied the parity constrained orientation problem in which the orientation must have a spanning arborescence rooted at a prescribed vertex. In his viewpoint, the problem arose in the different context of maximum genus embedding of graphs.

Frank, Jordán, and Szigeti [13] generalized this as follows. Given a function $m : V \to \mathbb{Z}_+$, an orientation with prescribed parity of out-degrees of a graph is to be reached for which there exist $m(V)$ edge-disjoint spanning arborescences s.t. exactly $m(v)$ of the arborescences are rooted at v. Then, $p(X) = m(V - X)$ if $X \neq \emptyset$.

One of the most general result in the area is due to Király and Szabó. They gave a characterization for the case when p is a non-negative intersecting supermodular function. Surprisingly, the parity involving modification (6.7) of the partition formula is sufficient in this case. If there is any characterization in a parity problem it is a rather uncommon occasion that the answer is a simple partition type formula.

Theorem 6.2.1 (Király and Szabó [27]). *Let $H = (V, E)$ be a hypergraph, let $p : 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$, and finally $T \subseteq V$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if the partition formula (6.7) holds for every partition $X_1, X_2, \ldots, X_t$ of V.*

Theorem 6.2.1 is unlikely to fit into the framework of solid polymatroids. In the previous sections we have seen various examples for NTCDCs in various polymatroids, we can see more in [33] in the polymatroid parity formulation of Mader's $\mathcal{A}$-paths problem. For the polymatroids arising in the orientation problem of Theorem 6.2.1 we do not show any. The pure explanation is that there are no NTCDCs but the contractions do have. This observation will lead to an elegant new proof of Theorem 6.2.1 in the Chapter 7. But at this point, our theory is not strong enough enough to prove the general case, however we can give a proof to Frank, Jordán, and Szigeti's Theorem.

6.2.1 Rooted connected orientations

Now, let us specialize our setup to the case when $p : 2^V \to \mathbb{Z}_+$,

$$p(X) = \begin{cases} m(V - X), & \text{if } \emptyset \neq X \subseteq V, \\ 0, & \text{otherwise.} \end{cases}$$

We deal with the technically bit more general problem which was introduced in Chapter 4. We repeat the construction. Let $S = V \cup \{uv : u, v \in V,\ u \neq v\}$, $S[X] = X \cup \{uv : u, v \in X,\ u \neq v\}$ and $X \subseteq V$, and $\mathcal{L} = \{S[X] : X \subseteq V\}$. Let $H = (V, E)$ be a hypergraph ($|e| \geq 1$ for every $e \in E$), $\alpha : V \to \mathbb{Z}_+$, $l \in \mathbb{Z}_+$, and finally,

$$b(U) = \begin{cases} \alpha(X) - |E[X]| - l, & \text{if } U = S[X] \text{ for some } \emptyset \neq X \subseteq V, \\ 0, & \text{if } U = \emptyset. \end{cases}$$

We always assume that α, H, and l are chosen so that $b \geq 0$. Then, $\widehat{b}$ is solid, as it has been stated in Chapter 4. Then, Corollary 4.0.4 characterizes $\nu(\widehat{b}|_{2^A})$ for any $A \subseteq S$.

We show moreover that if $\alpha \geq \sum_{e \in E} e$ i.e. H is "relatively small" as compared to α, then $\nu(\widehat{b}|_{2^V})$ can be characterized by the partition formula (2.3). The condition $\alpha \geq \sum_{e \in E} e$ will hold for the orientation problem, and we have to compute exactly $\nu(\widehat{b}|_{2^V})$ there.

Theorem 6.2.2. *Let b be as above and suppose moreover $\alpha \geq \sum_{e \in E} e$. Then,*

$$\nu(\widehat{b}|_{2^V}) = \min \sum_{j=1}^{s} \left\lfloor \frac{b(S[X_j])}{2} \right\rfloor,$$

where the minimum is taken over all partitions $X_1, X_2, \ldots, X_s$ of V.

Proof. We have to prove only the $\geq$ direction. For two disjoint sets $X, Y \subseteq V$, let $E[X, Y] = E[X \cup Y] - E[X] - E[Y]$. By Corollary 4.0.4, there exist $Z \subseteq S$ and $U_1, U_2, \ldots, U_t \in \mathfrak{L}_Z - \{\emptyset\}$ s.t. $V \subseteq \bigcup_{j=1}^{t} U_j$, and

$$\nu(\widehat{b}|_{2^V}) = \widehat{b}(Z) + \sum_{j=1}^{t} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor. \tag{6.9}$$

Let us choose the dual solution so that $\widehat{b}(Z)$ is as small as possible, and let t be as small as possible with respect to the primary conditions.

Lemma 6.2.3. *Then, $\widehat{b}(Z) = 0$.*

Proof. For $U \subseteq S$, let $\mathfrak{X}_U = \{X \subseteq V : S[X] \in \mathfrak{F}_U\}$, and let $X_j \subseteq V$ s.t. $U_j = S[X_j]$. Let $Z_1 \in \mathfrak{X}_Z$ with $b(S[Z_1]) > 0$, and let $1, 2, \ldots, t'$ be those indices j having $Z_1 \subseteq X_j$. Then, we have $t' \geq 2$, and $X_j \cap X_{j'} = Z_1$ for $j, j' \in [t']$, $j \neq j'$.

Let $Z' = Z - S[Z_1]$, and $U'_j = S[X_j - Z_1]$ for $j \in [t']$. Let $1, 2, \ldots, t''$ be those indices j for which $l - |E[X_j - Z_1, Z_1]|$ is even. Next, let $t'' \leq t''' \leq t'$, the exact value of t''' will be fixed later. In what follows, we show that if we replace Z by

$Z - S[Z_1]$, replace U_j by U'_j for each $j \in [t''']$, remove U_j for every $t''' + 1 \le j \le t'$, and introduce a new class $S[Z_1] \cup \bigcup_{j=t'''+1}^{t'} U_j$, then the right hand side of (6.9) does not increase, which contradicts the minimality of $\widehat{b}(Z)$. The whole thing is a bit messy but simple technical computation. First, by definition, we have

$$b_Z(U_j) = b_{Z'}(U'_j) + l - |E[X_j - Z_1, Z_1]|.$$

Next, by using the parity of $l - |E[X_j - Z_1, Z_1]|$, we get

$$\sum_{j=1}^{t'''} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \ge \sum_{j=1}^{t'''} \left(\left\lfloor \frac{b_{Z'}(U'_j)}{2} \right\rfloor + \frac{l - |E[X_j - Z_1, Z_1]|}{2} \right) - \frac{t''' - t''}{2},$$

and

$$\sum_{j=t'''+1}^{t'} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \ge \sum_{j=t'''+1}^{t'} \frac{b_Z(U_j) - 1}{2} \ge \frac{b_{Z'}(S[Z_1] \cup \bigcup_{j=t'''+1}^{t'} U_j) - b_{Z'}(S[Z_1]) - (t' - t''')}{2}$$

by submodularity. By $\alpha \ge \sum_{e \in E} e$, we have $b(S[Z_1]) \ge \sum_{j=1}^{t'} |E[X_j - Z_1, Z_1]| - l$.

If $l = 0$, then by $b(S[Z_1]) \ge \sum_{j=1}^{t'''} |E[X_j - Z_1, Z_1]| + (t' - t''')$, we get

$$b(S[Z_1]) + \sum_{j=1}^{t'} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \ge \sum_{j=1}^{t'''} \left\lfloor \frac{b_{Z'}(U'_j)}{2} \right\rfloor + \frac{b_{Z'}(S[Z_1] \cup \bigcup_{j=t'''+1}^{t'} U_j) - t''' + t''}{2},$$

thus $t''' = t''$ is a good choice.

If $l > 0$, then $b(S[Z_1]) \ge \sum_{j=1}^{t'''} |E[X_j - Z_1, Z_1]| - l$, and

$$b(S[Z_1]) + \sum_{j=1}^{t'} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \ge \sum_{j=1}^{t'''} \left\lfloor \frac{b_{Z'}(U'_j)}{2} \right\rfloor + \frac{b_{Z'}(S[Z_1] \cup \bigcup_{j=t'''+1}^{t'} U_j) - l + t'''l + t'' - t'}{2}.$$

By choosing $t''' = t'$ we get $-l + t'''l + t'' - t' = (t' - 1)(l - 1) - 1 + t'' \ge -1$, and hence

$$b(S[Z_1]) + \sum_{j=1}^{t'} \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \ge \sum_{j=1}^{t'''} \left\lfloor \frac{b_{Z'}(U'_j)}{2} \right\rfloor + \left\lfloor \frac{b_{Z'}(S[Z_1] \cup \bigcup_{j=t'''+1}^{t'} U_j)}{2} \right\rfloor - \frac{1}{2}.$$

As the other terms are integer, the $-\frac{1}{2}$ can be dropped. □

□

The partition type characterization of Frank, Jordán, and Szigeti [13] can be derived from Theorem 6.2.2 by simple computation:

Theorem 6.2.4 (Frank, Jordán, and Szigeti [13]). *Let $H = (V, E)$ be a hypergraph, $m : V \longrightarrow \mathbb{Z}_+$, $p : 2^V \longrightarrow \mathbb{Z}_+$, $p(X) = m(V - X)$ for $X \neq \emptyset$, $p(\emptyset) = 0$, and finally $T \subseteq V$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if the partition formula (6.7) holds for every partition $X_1, X_2, \ldots, X_t$ of V.*

Proof. It is not hard to see that we can restrict ourselves to the case $T = \emptyset$. Then (6.7) transforms to

$$g(V) \leq 2 \sum_{j=1}^{t} \left\lfloor \frac{g(X_j)}{2} \right\rfloor . \tag{6.10}$$

Let $\alpha = m + \sum_{e \in E} e$ and $l = m(V)$. We prove that (6.10) implies $2\nu(\widehat{g}) = g(V)$, and therefore the existence of an orientation covering p having moreover even out-degrees. If $\emptyset \neq X \subseteq V$, then $\sum_{e \in E} e(X) \geq |E| - |E[V - X]|$. Then, by applying (6.10) to the partition $\{X, V - X\}$ we get

$$\sum_{e \in E} e(X) - |E[X]| \geq |E| - |E[V - X]| - |E[X]| \geq m(X) + m(V - X),$$

which gives $g \geq 0$.

By Theorem 6.2.2, there exists a partition $X_1, X_2, \ldots, X_s$ with

$$\nu(\widehat{g}) = \sum_{j=1}^{s} \left\lfloor \frac{g(X_j)}{2} \right\rfloor ,$$

and the statement follows. □

The problem can be put in a bit more general setting, namely we can also prescribe lower bounds on the out-degrees. By a note of Chapter 4, this case reduces to the matching problem of a solid polymatroid.

6.3 Crossing supermodular orientations

We have already seen some orientation problems with crossing supermodular connectivity requirement. The problem is of this kind when we have lower and upper bounds on the out-degrees, another one is the rooted problem with lower bounds on the out-degrees. Now we switch to a crossing supermodular case when p describes a non-trivial connectivity requirement. However, very little is known about these cases, we usually do not now combinatorial characterizations to the parity constrained case. The only case for which we have some partial result is the case of strongly connected requirement.

6.3.1 Strongly connected orientation of planar graphs

In one of the most simple orientation problems with crossing supermodular connectivity we have $p(X) = 1$ if $\emptyset \neq X \subsetneq V$, i.e. we are looking for a strongly connected orientation (*strong orientation* for short). We consider only the graph case, we assume for sake of simplicity that the considered graph $G = (V, E)$ is connected, it has no loop, and $T = \emptyset$ i.e. we prescribe even out-degree for each vertex. A strong orientation having even out-degrees is referred as *even strong orientation*. It is well-known that G has a strong orientation if and only if G is 2-edge-connected. As we are interested in the existence of a more restricted configuration, let us assume that G is 2-edge-connected.

The out-degree vectors of the strong orientations of G are exactly the integer solutions of the system

$$P = \{x \in \mathbb{R}^V_+ : x(V) = |E|,\ x(X) \leq |e(X)| - 1 \text{ for every } \emptyset \neq X \subsetneq V\} =$$
$$\{x \in \mathbb{R}^V_+ : x(V) = |E|,\ x(X) \leq |e(X)| - c(G[V - X]) \text{ for every } \emptyset \neq X \subseteq V\},$$

where for a graph G', $c(G')$ denotes the number of its components, and (only in this section) $e(X)$ denotes the set of edges having at least one end-vertex in X. It can be seen easily that P is a base polyhedron. Therefore, G admits an even strong orientation if and only if P contains an even vector. Moreover,

$$|e(X)| - c(G[V - X]) = |e(X)| - c((V, E[V - X])) + 1 + |X| - 1 =$$
$$r_{\mathcal{M}^*}(e(X)) + |X| - 1,$$

where $\mathcal{M}$ is the cycle matroid of G and $\mathcal{M}^*$ is its dual. Then, $h : 2^V \to \mathbb{Z}$,

$$h(X) = \begin{cases} r_{\mathcal{M}^*}(e(X)) + |X| - 1, & \text{if } \emptyset \neq X \subseteq V, \\ 0, & \text{if } X = \emptyset \end{cases}$$

is a non-negative non-decreasing intersecting submodular function, and $\widehat{h} : 2^V \to \mathbb{Z}_+$, $\widehat{h}(X) = \min_{X \subseteq X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_t} \sum_{i=1}^t h(X_i)$ is a polymatroid function. By this terminology, $P = \mathcal{P}(\widehat{h}) \cap \{x \in \mathbb{R}_+^V : x(V) = |E|\}$, and G has an even strong orientation if and only if

$$2\nu(\widehat{h}) = |E|. \tag{6.11}$$

The goal would be to give a combinatorial characterization to the existence of an even strong orientation, i.e. a characterization to (6.11) in terms of G. We were not able to solve this task in general. First we describe a linear representation of $\widehat{h}$, but we have to note that we are not aware of an algorithm which gives a representation in a deterministic way. Finally, for the case of planar graphs, we embed $\mathcal{P}(\widehat{h})$ into a small solid polymatroid.

Linear representation

First of all, recall that the dual of the graphic matroid is linear, let us represent it in $\mathbb{R}^k$ for some k, by associating the vector $y(e) \in \mathbb{R}^k$ with $e \in E$. The representation of $\widehat{h}$ will be in the space $\mathbb{R}^k \times \mathbb{R}^V$. Let $z(v)$ be the unit vector of $\mathbb{R}^V$ associated with v. For each vertex $v \in V$, we consider the subspace l_v of $\mathbb{R}^k \times \mathbb{R}^V$ generated by the vectors $\{(y(e), 0) : e \in \delta(v)\}$ and $(0, z(v))$. Let H be a hyperplane of $\mathbb{R}^k \times \mathbb{R}^V$ in general position (w.r.t. the above constructed subspaces l_v, $v \in V$). For more on this construction, see Lovász [32]. Then, simple computation shows that the subspaces $\{l_v \cap H : v \in V\}$ together represent $\widehat{h}$.

As in the case of (k, l)-matroids, this construction does not give a deterministic polynomial algorithm to construct a representation. Therefore the problem of the existence of an even strong orientation is random polynomial, we do not know whether it is in NP∩co-NP, however it is quite likely.

In the remaining part of the chapter, we show a combinatorial NP∩co-NP characterization for planar graphs.

Planar graphs

The key fact used here is that the dual graphic matroid of a planar graph is also a graphic matroid. Let us fix a planar embedding of G, let V^* be its set of faces and let G^* be the dual graph. Consequently, let e^* be the edge of G^* corresponding to $e \in E$. Hence, $F \subseteq E$ is independent in $\mathcal{M}^*$ if and only if $\{e^* : e \in F\}$ is independent in the cycle matroid of G^*. Let us repeat the above construction of the linear representation in a more combinatorial setting.

Let $S = \binom{V \cup V^*}{3}$, i.e. the set of triplets of $V \cup V^*$. For $X \subseteq V$, let $\Gamma^*(X) = \{v^* \in V^* : v^*u^* = e^* \text{ for some } e \in e(X)\}$, i.e. the set of faces incident to X. Let $\mathcal{L} = \{S[X \cup X^*] : X \subseteq V, X^* \subseteq V^*, \Gamma^*(X) \subseteq X^*\}$. Let $b : \mathcal{L} \to \mathbb{Z}_+$ s.t.

$$b(S[X \cup X^*]) = \begin{cases} |X \cup X^*| - 2, & \text{if } |X \cup X^*| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

For $v \in V$, let $S_v = S[\{v\} \cup \Gamma^*(\{v\})] = \{xyv \in S : e \in \delta_G(v),\ e^* = xy\}$. Clearly, $b(S_v) = |\{v\} \cup \Gamma^*(v)| - 2 = |\Gamma^*(v)| - 1 = r_{\mathcal{M}^*}(e(v))$ for any $v \in V$.

Claim 6.3.1. *Let $\emptyset \neq X \subseteq V$. Then,*

$$\min_{X = X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_t,\ X_i \neq \emptyset} \sum_{i=1}^{t} b(S[X_i \cup \Gamma^*(X_i)]) = \widehat{h}(X).$$

Proof. For the easier direction, let $X \subseteq V$ and $X^* \subseteq V^*$ s.t. $|X \cup X^*| \geq 2$ and $\Gamma^*(X) \subseteq X^*$. Then, $b(S[X \cup X^*]) = |X \cup X^*| - 2 = |X^*| - 1 + |X| - 1 \geq r^*_{\mathcal{M}}(e(X)) + |X| - 1 = h(X) \geq \widehat{h}(X)$.

For the other part, let $\emptyset \neq X \subseteq V$. Then, $h(X) = r_{\mathcal{M}^*}(e(X)) + |X| - 1 = \sum_{i=1}^{t}(|X_i^*| - 1) + |X| - 1$ for a subpartition $X_1^*, X_2^*, \ldots, X_t^*$ of V^* s.t. $\{e^* : e \in e(X)\} \subseteq \bigcup_{i=1}^{t} E^*[X_i^*]$. For each $v \in X$, there exists a unique i s.t. $\{e^* : e \in e(v)\} \subseteq E^*[X_i^*]$. Let $X_i = \{v \in X : \{e^* : e \in e(v)\} \subseteq E^*[X_i^*]\}$. Then, $\sum_{i=1}^{t}(|X_i^*| - 1) + |X| - 1 \geq \sum_{i=1}^{t}(|X_i^*| - 1 + |X_i| - 1) = \sum_{i=1}^{t} b(S[X_i^* \cup X_i])$. □

Hence, $\widehat{h}$ is a homomorphic map of the matroid rank function $\widehat{b}$. Next, we have to observe that $\widehat{b}$ is solid. The main result of this section is the following:

Theorem 6.3.2. *Let $G = (V, E)$ be a 2-edge-connected loopless planar graph. Then, G has an even strong orientation if and only if*

$$2\widehat{b}(Z) + \sum_{j=1}^{t} 2 \left\lfloor \frac{b_Z(U_j)}{2} \right\rfloor \geq |E| \tag{6.12}$$

for every $Z \subseteq S$ and for every family of sets $U_1, U_2, \dots, U_t \in \mathcal{L}_Z - \{\emptyset\}$ s.t. for each $v \in V$, there exists an U_j with $\bigcup_{v \in V} S_v \subseteq U_j$.

Let us consider an example. Let G be the cube, i.e. $V = \{0,1\}^3$ and $E = \{uv : |u \Delta v| = 1\}$, where Δ stands for the symmetric difference. Then, $\widehat{b}(S[V^*]) = b(S[V^*]) = 6 - 2 = 4$. For any $v \in V$ we have $b_{S[V^*]}(S_v) = b(S[\{v\} \cup V^*]) - \widehat{b}(S[V^*]) = (7-2) - 4 = 1$. Thus,

$$\nu(\widehat{h}) \leq \widehat{b}(S[V^*]) + \sum_{v \in V} \left\lfloor \frac{b_{S[V^*]}(S_v)}{2} \right\rfloor = 4 + \sum_{v \in V} \left\lfloor \frac{1}{2} \right\rfloor = 4.$$

As $|E| = 12$, this means that there is no even strong orientation. In fact, we have equality here, an orientation having 4 vertices of out-degree 2 and 4 vertices of out-degree 1 can be constructed easily. Figure 6.1 shows a NTCDC of $\widehat{h}$. If we

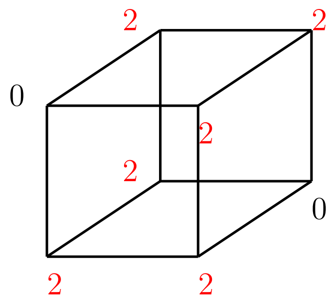

Figure 6.1: A NTCDC of $\widehat{h}$ in the cube

decrease any two of the positive values by one, and substitute the 0's by 1's, then there will be a strong orientation with these out-degrees. The 0's are at distance 3 from each other (in the graph), hence the two 2's to decrease are both at distance 1 from a 0, or at distance 1 from two different 0's. Hence, we have 3 different cases which are shown in Figure 6.2. And the intersection of the spans of the circuits has rank 0. If we do the extension described above, then the intersection will be $S[V^*]$ which has rank 4.

It might not be impossible to give a simpler embedding into a matroid having the DCP, but might be a hard task. It is more important to formalize Theorem 6.3.2 in a human comprehensible form. If this is not possible, then we cannot expect nice combinatorial description for the non-planar case.

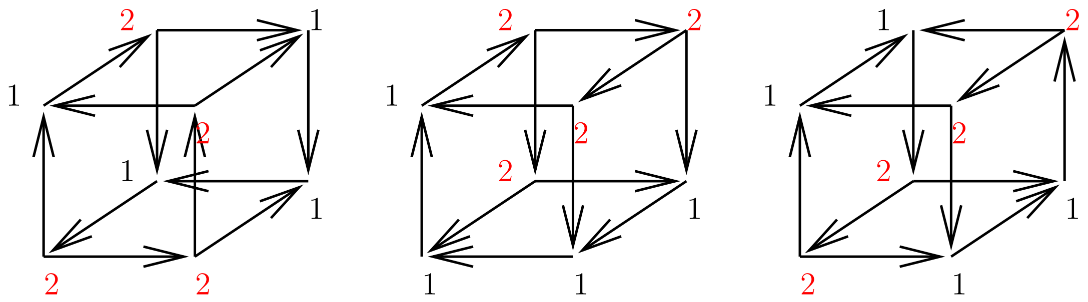

Figure 6.2: Orientations

Chapter 7

Polymatroids without NTCDCs

We have shown in Section 6.2 that Theorem 6.2.4 reduces to the DCP of solid polymatroids. The abstract definition of the connectivity requirement in Theorem 6.2.1 makes the same thing unlikely for the more general setting of Theorem 6.2.1. However, we think that there must be a more general reasoning in the background of Theorem 6.2.1, a more general one than its original inductive proof, in other words there must be a way to place this statement in polymatroid parity theory. The answer for this question was given by Szabó and the author in [43]. First of all we observed that the polymatroids formed by the out-degree vectors of the orientations meeting these connectivity requirements have no NTCDCs at all, however the contraction used in the inductive step of Theorem 2.6.1 destroys this property. Finally, we have shown how to get rid of the contraction, and proved that a partition type characterization holds.

This chapter is dedicated to the polymatroids having no NTCDCs. Most of these results are from [43] which is a joint work with Jácint Szabó. As a starting point we recall an immediate consequence of (the proof of) Theorem 2.3.1:

Theorem 7.0.3 (Lovász [33]). *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function. Suppose that for every $s \in S$ there exists a maximum matching m s.t. $s \notin \mathrm{sp}_f(m)$. If f has no even NTCDCs, then*

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor, \tag{7.1}$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of S.

If the polymatroid has no NTCDCs, then the condition that every $s \in S$ is avoided by the span of a maximum matching can be dropped. Let us emphasize that this is a surprisingly rare phenomenon. We know only a few examples in which a partition type formula characterizes the size of the maximum matching.

Theorem 7.0.4. *([43]) Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function without NTCDCs. Then,*

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of S.

Corollary 7.0.5. *([43]) Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function without NTCDCs, and let $T \subseteq S$. Then, we have*

$$\delta_T(f) = f(S) - \min \left(\sum_{j=1}^{t} f(U_j) - |\{j : f(U_j) \not\equiv |T \cap U_j|\}| \right),$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of S.

Our aim in the present chapter is to explore how some polymatroid operations alter the double circuits, or keep the property that a certain type of double circuit does not exists. Thereafter, we will show examples for polymatroids without NTCDCs.

We have seen in Claim 2.6.5 that under some circumstances the NTCDCs of a contraction correspond to NTCDCs of the original matroid. The counterexample presented after the statement shows that the contraction can destroy the property of having no NTCDCs. Hence, contractions must be avoided in the proof of Theorem 7.0.4, while the application of other operations may have fruitful properties.

(7.2i) **Translation.**

Claim 7.0.6. *If $n \in \mathbb{Z}^S$ and f, $f + n$ are polymatroid functions, then a vector w is a double circuit of f with $U = \operatorname{supp}(w)$ if and only if $w + n|_U$ is a double circuit of $f + n$. In this case their principal partitions coincide.*

Proof. Clearly, $r_{f+n}(x + n) - (x + n)(S) = r_f(x) - x(S)$ for all $x \in \mathbb{Z}^S$. Thus by symmetry, it is enough to prove that if w is a double circuit of f with support W then $w_s + n_s > 0$ for every $s \in U$. Otherwise, if $w_s + n_s \leq 0$, then we would have

$$w(U - s) \geq w(U) + n_s = f(U) + 2 + n_s \geq f(U - s) + 2,$$

which is impossible. □

(7.2ii) **Deletion or upper bound.**

Claim 7.0.7. *Let $u \in \mathbb{Z}_+^S$. If $w \in \mathbb{Z}_+^S$ is a double circuit of $f' := f \backslash u$ then w is either a double circuit of f with the same principal partition, or w is trivial or non-compatible w.r.t. f'.*

Proof. If $w \leq u$ then w is a double circuit of f with the same principal partition. Observe that $w_s \leq f'(\{s\}) + 2$ and $f'(\{s\}) \leq u_s$ for every $s \in S$. Thus if $w \not\leq u$ then there exists an $s \in S$ such that $w_s - f'(\{s\}) \in \{1, 2\}$. If $w_s = f'(\{s\}) + 2$ then $r_{f'}(w_s \chi_s) = w_s - 2$, thus w is non-compatible. If $w_s = f'(\{s\}) + 1$ then $w_s \chi_s$ is a circuit of f' thus w is either non-compatible, or if w is compatible then it is trivial. □

(7.2iii) **Direct sum.** The double circuits of $f_1 \oplus f_2$ are exactly the double circuits of f_1 and f_2.

(7.2iv) **Dual.** Let $S = \{1, 2, 3\}$, and let f be the rank function of the uniform matroid $U_{3,2}$ on S, i.e. $f(X) = \min\{2, |X|\}$. Then, $f_{\mathbf{1}}^*(X) = 1$ for each non-empty set X, i.e. $f_{\mathbf{1}}^*$ is the rank function of $U_{3,1}$. It can be seen that f has no compatible double circuit, while $(1, 1, 1)$ is a compatible double circuit of $f_{\mathbf{1}}^*$. However, not everything is lost, since

$$\begin{aligned}\delta_T(f^*) &= \min\{|\{s \in S : x_s \not\equiv \chi_T(s)\}| : x \in B(f_u^*)\} = \\ &\min\{|\{s \in S : u_s - y_s \not\equiv \chi_T(s)\}| : y \in B(f)\} = \delta_{\{s \in S : \chi_T(s) \not\equiv u_s\}}(f).\end{aligned}$$

Hence, dualization is fruitful if $\delta_{\{s \in S : \chi_T(s) \not\equiv u_s\}}(f)$ is tractable by some reason. It may also happen that we can reach a NTCDC-free polymatroid by dualization, this is the case for a possible polymatroid parity formulation of the ordinary matching problem of graphs.

(7.2v) **Truncation.** This operation can create new NTCDCs. Let $|S| = 3$ and $f = |\cdot|$. Now all double circuits of f are of the form $3\chi_s$, while f^1 admits a NTCDC $(1, 1, 1)$ with $d = 3$.

(7.2vi) **Dilworth Truncation.** The Dilworth truncation can create new compatible non-trivial double circuits. Let $S = \{1, 2, 3\}$, and $f(X) = |X|+2$, if X is not empty. If x is a NTCDC of f, then its principal partition partitions S into singletons, $x(\{1, 2\}) = x(\{1, 3\}) = x(\{2, 3\}) = 4$, i.e. $x(S) = 6$ which contradicts to $x(S) = 7$. Hence, f has no NTCDCs, while the Dilworth-truncation of f has the compatible double circuit $(2, 2, 2)$.

We do not know how homomorphic image and sum alter double circuits. The most simple question would be, whether $f_1 + f_2$ has NTCDCs if f_1 and f_2 do not have any. Now, we are ready to prove Theorem 7.0.4.

Proof of Theorem 7.0.4. The first part of the statement is proved by induction on $|K(f)|$, where

$$K(f) = \{s \in S : s \in \mathrm{sp}_f(m) \text{ for each maximum matching } m\}.$$

Case 1. If $K(f) = \emptyset$, then we are done by Theorem 7.0.3.

Case 2. Next, let $K(f) \neq \emptyset$. If m is a maximum matching of $f + 2\chi_s$, then $m(s) \geq 2$. Indeed, let $m(s) = 0$. As m is a maximum matching, there exists a set $s \in U \subseteq S$ with $m(U) \geq (f + 2\chi_s)(U) - 1$. This means $m(U - s) = m(U) \geq (f + 2\chi_s)(U) - 1 \geq f(U - s) + 1$, which is a contradiction.

It is also clear that $m + 2\chi_s$ is a matching of $f + 2\chi_s$ for each matching m of f. Therefore, m is a maximum matching of f if and only if $m + 2\chi_s$ is a maximum matching of $f + 2\chi_s$.

Let $s \in K(f)$. Then, $\nu(f) \leq \nu(f + \chi_s) \leq \nu(f + 2\chi_s) = \nu(f) + 1$. We claim that $\nu(f + \chi_s) = \nu(f)$. Indeed, if m is a maximum matching of $f + \chi_s$ and $\nu(f + \chi_s) = \nu(f) + 1$, then m is also a maximum matching of $f + 2\chi_s$, proving $m(s) \geq 2$. Then, $m - 2\chi_s$ is a maximum matching of f, and there exists a set $s \in U \subseteq S$ with $(m - 2\chi_s)(U) = f(U)$. This implies $m(U) = f(U) + 2$ which contradicts that m is a matching of $f + \chi_s$.

Moreover, this gives $K(f + \chi_s) \subsetneq K(f)$. By Lemma 7.0.6, $f + \chi_s$ has no NTCDC, and we can apply induction to $f + \chi_s$. This gives a partition $U_1, U_2, \ldots, U_t$

of S s.t.

$$\nu(f+\chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{(f+\chi_s)(U_j)}{2} \right\rfloor .$$

But then,

$$\nu(f) = \nu(f+\chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{(f+\chi_s)(U_j)}{2} \right\rfloor \geq \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor .$$

□

Proof of Corollary 7.0.5. As $\delta_T(f) = \delta_\emptyset(f+\chi_T)$, and the translation does not create new NTCDCs, we can apply Theorem 7.0.4 to $f+\chi_T$.

$$\delta_T(f) = \delta_\emptyset(f+\chi_T) = (f+\chi_T)(S) - 2\nu(f+\chi_T) =$$
$$f(S) + |T| - 2 \min_{U=U_1\dot\cup U_1\dot\cup\ldots\dot\cup U_t} \sum_{j=1}^{t} \left\lfloor \frac{(f+\chi_T)(U_j)}{2} \right\rfloor ,$$

which is exactly what we have stated. □

We know only one nontrivial class of polymatroids without NTCDCs. It is a great fortune that these polymatroids appear as degree vectors of connectivity orientation problems. Of course, this is not an accident, the first goal of examining NTCDC-free polymatroids was the better comprehension of the orientations with intersecting supermodular requirement. In fact,

Lemma 7.0.8. *Let $p: 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function s.t. $p(\emptyset) = 0$, and*

$$\binom{d-1}{2} p(U) < \sum_{1\leq i<j\leq d} p(U-U_i-U_j) + 2\binom{d-1}{2}, \tag{7.3}$$

whenever $U \subseteq V$ and $U_1, U_2, \ldots, U_d$, $d \geq 3$ is a partition of U into non-empty sets. Let $\alpha: V \to \mathbb{Z}_+$, $H = (V,E)$ be a hypergraph, and suppose that the function $f: 2^V \to \mathbb{Z}$ defined by $f(U) = \alpha(U) - |E[U]| - p(U)$ is non-negative and non-decreasing. Then, $\widehat{f}$ has no NTCDC.

Hence, for (7.3), p can be constant on non-empty sets, or modular, or of form $p(U) = |E'[V-U]|$ $(U \neq \emptyset)$ for some hypergraph (V, E'), or any $0-1$ valued non-negative intersecting supermodular function, or any non-negative intersecting supermodular function which is non-increasing on non-empty sets.

The example presented after Claim 2.6.5 is of this type, by the choice $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_i, v_iv_i : i \in \{2,3,4\}\}$, $p(\{v_1\}) = 1$ and $p(U) = 0$ for the other sets, and $\alpha = \sum_{e \in E} e$.

Observe moreover that these f's are generalizations of the polymatroid functions considered in Theorem 6.2.2.

Proof. For contradiction, let $x : V \to \mathbb{Z}_+$ be a NTCDC of $\widehat{f}$ with principal partition $U = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_d$. Then,

$$x(U) \leq f(U) + 2.$$

Let $1 \leq i < j \leq d$. As $x|_{U-U_i}$ and $x|_{U-U_j}$ are circuits, we have

$$x(U - U_i) = f(U - U_i) + 1, \quad \text{and} \quad x(U - U_j) = f(U - U_j) + 1.$$

As $x|_{U-U_i-U_j}$ is independent,

$$c_{i,j} + x(U - U_i - U_j) = f(U - U_i - U_j)$$

for a non-negative integer $c_{i,j}$. By applying submodularity to $U - U_i$ and $U - U_j$, we get

$$c_{i,j} = f(U) - f(U - U_i) - f(U - U_j) + f(U - U_i - U_j) \leq 0,$$

Therefore, $U - U_i$ and $U - U_j$ is a modular pair of sets, i.e., they satisfy the submodular inequality with equality, and $c_{i,j} = 0$. This means, that each hyperedge $e \in E[U]$ is spanned by one of the U_i's. Therefore, by the modularity of these pairs, we have

$$\binom{d-1}{2}(f(U)+2) = \binom{d-1}{2}x(U) = \sum_{1 \leq i < j \leq d} x(U - U_i - U_j) = \sum_{1 \leq i < j \leq d} f(U - U_i - U_j). \tag{7.4}$$

On the other hand,

$$\binom{d-1}{2}\alpha(U) = \sum_{1 \leq i < j \leq d} \alpha(U - U_i - U_j),$$

as α is modular. Last,

$$\binom{d-1}{2}|E[U]| = \binom{d-1}{2}\sum_{i=1}^{d}|E[U_i]| = \sum_{1\le i<j\le d}|E[U-U_i-U_j]|.$$

Using the definition of f, we get that the last 2 inequalities and (7.3) together contradict (7.4). $\square$

7.1 Intersecting supermodular orientations again

Finally, we are ready to prove Theorem 6.2.1, based on polymatroids with no NTCDCs:

Proof of Theorem 6.2.1. If $X_1, X_2, \ldots, X_t$ is a partition of V, then the necessity of condition (6.7) is clear. Hence, we have to prove that if (6.7) holds for every partition of V, then H has an orientation covering p with odd out-degrees exactly in the vertices of T.

Let $\alpha : V \to \mathbb{Z}_+$ s.t. $\alpha(v) = \sum_{e\in E} e(v)$. Then $g(U) = \alpha(U) - |E[U]| - p(U)$.

Claim 7.1.1. *g is non-negative and non-decreasing.*

Proof. If $\emptyset \neq U \subsetneq V$, then by (6.7) we have $g(U)+g(V-U) \ge g(V)$, and therefore

$$g(U) \ge \alpha(V) - |E| - \alpha(V-U) + |E[V-U]| + p(V-U) \ge \\ \alpha(U) - |E| + |E[V-U]| = \alpha(U) - |\{e \in E : e \cap U \neq \emptyset\}| \ge 0,$$

and $g(V) = \alpha(V) - |E| \ge 0$ holds trivially. If $v \in U \subseteq V$, $|U| \ge 2$, then

$$g(U) - g(U-v) = \alpha(v) - |E[U]| + |E[U-v]| - p(U) + p(U-v) \ge \\ \alpha(v) - |\{e \in E[U] : v \in e\}| \ge 0.$$

$\square$

Therefore, α, H, p, and g are as in Lemma 7.0.8, and we can apply Theorem 7.0.4 to $\widehat{g}$. This gives that there exists a partition $U_1, U_2, \ldots, U_t$ of V s.t.

$$\delta_T(\widehat{g}) = \widehat{g}(V) - \left(\sum_{j=1}^{t}\widehat{g}(U_j) - |\{j : \widehat{g}(U_j) \not\equiv |T\cap U_j|\}|\right).$$

By the definition of $\widehat{g}$, we get that there exists a partition $U'_1, U'_2, \ldots, U'_{t'}$ of V s.t.

$$\delta_T(\widehat{g}) \leq \widehat{g}(V) - \left(\sum_{j=1}^{t'} g(U'_j) - |\{j : g(U'_j) \not\equiv |T \cap U'_j|\}| \right) \leq \widehat{g}(V) - g(V) \leq 0,$$

which completes the proof. □

Theorem 6.2.1 has also a defect form, which can be proved similarly:

Theorem 7.1.2 (Király and Szabó [27]). *Let $H = (V, E)$ be a hypergraph, $T \subseteq V$, $p : 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$, and assume that H has an orientation covering p. Define b as in Theorem 6.2.1. For an orientation D of H let $Y_D \subseteq V$ denote the set of odd out-degree vertices in D. Then*

$$\min \{|T \triangle Y_D| \ : \ D \text{ is an orientation of } H \text{ covering } p\} = \max \left\{ b(V) - \sum_{j=1}^{t} b(U_j) + |\{j : b(U_j) \not\equiv |T \cap U_j|\}| \right\},$$

where the maximum is taken on partitions $U_1, U_2, \ldots, U_t$ of V.

7.2 Graph matching

It is shown in [27] that Theorem 6.2.1 contains the question whether an undirected graph has a perfect matching, even in the special case when p can be positive only on singletons. Let therefore $G = (V, E)$ be an undirected graph, and $q : 2^E \to \mathbb{Z}_+$ as in Section 2.1. Then, q has NTCDCs, moreover we will encounter NTCDCs in most of the natural formulations of the matching problem. The resolution of the seeming contradiction is that there are formulations without NTCDCs. By dualizing q w.r.t. the **2** vector, we get

$$q^*_{\mathbf{2}}(F) = 2|F| + |\bigcup(E - F)| - |\bigcup E| = 2|F| - |\{v \in V : e(v) \subseteq F\}|, \tag{7.5}$$

for $F \subseteq E$, where $e(v)$ denotes the set of edges adjacent to v. By considering the the transpose of the hypergraph G, then this can be rewritten as

$$q^*_{\mathbf{2}}(F) = 2|F| - |V[F]|. \tag{7.6}$$

This is a polymatroid considered by Lemma 7.0.8, and $\nu(q^*_{\mathbf{2}})$ is characterized by a partition formula.

7.3 A pinning down problem

Let $G = (V, E)$ be a graph whose vertices are points of the Euclidean plane. Let $p : V \to \mathbb{R}^2$, let us position vertex v into the point $p(v)$, the edges of G are rigid bars and G has flexible joins at the vertices. An *infinitesimal motion* means an assignment of *velocity* $x(v)$ to each vertex v s.t. the bar lengths are preserved. The *framework* (G, p) is called *rigid* if all the infinitesimal motions of (G, p) correspond to isometries of the plane. By *pinning down* a vertex v, we mean of fixing $x(v)$ to 0. The question of pinning down the minimum number of vertices resulting a rigid framework was solved by Lovász [33] in his seminal paper about matroid parity. We say that G is *generically rigid* if each framework (G, p) with *algebraically independent* coordinates are rigid. (In fact, G is generically rigid, if (G, p) is rigid for any one of these frameworks.) Then, generic rigidity is the property of the graph. For the generic case, instead of pinning down a minimum set of vertices it is better to say that we are looking for the smallest complete graph that adding it results in a generically rigid one. The problem was solved by Zs. Fekete [12]. The setup of [12] puts the problem into a bit more general framework. There, the problem is examined for $\mathcal{M}_{2,l}$, $l \in \{2, 3\}$.

For $Z \subseteq V$, let K_Z be the graph with vertex set Z having $4 - l$ parallel edges between any two vertices of Z. By assuming $r(E) < 2|V| - l$, we can ask for the minimum $Z \subseteq V$ s.t. $G + K_Z$ has rank $2|V| - l$. For $l = 2$, this is equivalent to the problem of shrinking a minimum vertex set Z s.t. G/Z has two edge-disjoint spanning trees. Loosely speaking, for $l = 3$, this is equivalent of pinning down a minimum set Z, s.t. the resulting graph is generically rigid. We may assume that E is independent in $\mathcal{M}_{2,l}$, since if E is replaced by one of its maximum independent sets, then the solution set does not change. For $X \subseteq V$, let $e(X)$ be the set of edges having at least one vertex in X. Fekete proved the following [12].

Lemma 7.3.1. *Let $l = 2, 3$, E independent in $\mathcal{M}_{2,l}$, and $r(E) < 2|V| - l$. If $Z \subseteq V$ and $|Z| \geq 2$, then*

(7.7i) $r(G + K_Z) = \min_{X \supseteq Z}(2|X| - l + |e(V - X)|)$,

(7.7ii) $r(G + K_Z) = 2|V| - l$ *if and only if* $|e(Y)| \geq 2|Y|$ *for every* $Y \subseteq V - Z$.

Therefore, the goal is to maximize the size of the set $V - Z$ s.t. $|e(Y)| \geq 2|Y|$ for every $Y \subseteq V - Z$. Let $f : 2^V \to \mathbb{Z}_+$ be the polymatroid function s.t.

$f(X) = \min_{X_1 \dot\cup X_2 = X}(|e(X_1)| + 2|X_2|)$, i.e. the polymatroid function obtained from $X \mapsto |e(X)|$ by putting upper bound 2 on the coordinates (defined previously as deletion w.r.t. the vector $\mathbf{2}$). Hence, for $l = 2$, the minimum cardinality set Z to contract has size $|V| - \nu(f)$, and for $l = 3$, the minimum set to pin down has size $|V| - \nu(f)$. It is shown in [12] that the computation of $\nu(f)$ reduces to the maximum matching problem of graphs. Therefore, $\nu(f)$ can be computed very efficiently. However, it is not hard to prove from our previous results that $\nu(f)$ has a partition type characterization. First, by Lemma 7.0.8, the polymatroid function $X \mapsto |e(X)|$ has no NTCDCs. As f is obtained from $X \mapsto |e(X)|$ by deletion, Claim 7.0.7 yields that f does not have one either. Thus,

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of V. By the definition of f, we get the following.

Theorem 7.3.2 (Fekete, [12])**.**

$$\nu(f) = \min \left(\left| V - \bigcup_{j=1}^{t} U_j \right| + \sum_{j=1}^{t} \left\lfloor \frac{e(U_j)}{2} \right\rfloor \right),$$

where the minimum is taken over all subpartitions $U_1, U_2, \ldots, U_t$ of V.

This answers the question of the case $l = 2$ as well as of the case $l = 3$. It is a seeming contradiction that we get the same answer for both cases. However, for being E independent in $\mathcal{M}_{2,l}$ we left out a different number of edges from G in the different cases.

7.4 A note on the weighted case

The solution of the weighted polymatroid parity problem is a long standing open question for any case which does not reduce to the weighted matching problem of graphs or to the weighted matroid intersection problem. The characterization is messy for linear or for DCP polymatroids even in the cardinality case. Hence, we might think that the class of NTCDC-free polymatroids is a good candidate to have a simple characterization in the weighted case. A natural question would be that the system

$$x \in \mathbb{R}^S,\ x \geq 0, \tag{7.8}$$

$$x(U) \leq \left\lfloor \frac{f(U)}{2} \right\rfloor \quad \text{for every } U \subseteq S \tag{7.9}$$

describes the convex hull of matchings of f if f is NTCDC-free. However, this is not true. The smallest example for this is the polymatroid f on ground set $S = \{v_1, v_2, v_3, v_4\}$, with $f(U) = 2|U|$ if $|U| \leq 1$, $f(U) = 3$ if $|U| = 2$ and $v_1 \in U$, and $f(U) = 4$ otherwise. Then, f has no NTCDCs and $(1/2, 1/2, 1/2, 1/2)$ is a vertex of (7.8-7.9). The example presented after Claim 2.6.5 is also good for this purpose. In that case $(1, 1/2, 1/2, 1/2)$ will be a vertex.

Chapter 8

Open questions

8.1 Relation of matroid classes

There are some structural questions concerning the relation of (poly)matroid classes presented in the dissertation which are left open. Figure 8.1 shows relations between (poly)matroid classes and parity problems of particular (poly)matroids. Adjectives indicating fullness or density are not indicated. We hope that it does not cause confusions that we put matroid classes or properties in some boxes, polymatroid classes in others, and optimization problems into the rest. The reader is asked to pass over this inaccuracy.

On the upper and left part we see abstract properties, while particular (poly)matroid classes and problems on the right and lower part. From the theoretical point of view it would be impressive to discover further relations, between the abstract properties. Say, we do not know the relation of pseudomodularity to solidness or to (k, l)-matroids. We do not know the relation of DCP and MDCP matroids, which however does not seem to be an important question as all the known constructions for DCP matroids are in fact MDCP.

A more interesting problem is the existence of a reasonable common generalization of MDCP matroids and NTCDC-free polymatroids. This would also have algorithmic interest. NTCDC-free polymatroids do not have structural properties which would imply say linearity, but an example showing this statement would help in completing the picture.

8.2 Algorithmic aspects

This short section is dedicated to mention some special cases of the above polymatroid classes where the known polynomial algorithms can be applied to compute a maximum matching and a combinatorial dual characterization.

As the matroid parity problem is more general than matroid intersection or graph matching, an algorithmic point of view to the problem needs more involved techniques and ideas. The very first algorithm, Lovász's one [35; 39] manipulates with maximum matchings and the dual solution is derived from the interaction of all maximum matchings. It is based on Theorem 2.3.1, and proves Theorem 2.5.1 in an algorithmic way. There are faster algorithms for the same problem presented by Orlin and Vande Vate [45] and by Gabow and Stallmann [17].

We essentially have the family of DCP and NTCDC-free polymatroids, or some special subclasses which have to be handled. Lovász algorithm can be used for some DCP polymatroids. This algorithm is presented essentially in two different ways, by Lovász [35; 39] and by Schrijver [51]. Both uses the DCP, and also some extra property. In Lovász' version, a principal extension is used at most once during the algorithm, an then the added element is contracted. Schrijver's version relies on the modularity of flats. Both can be used in vector spaces (over a large enough field), but any of them hardly in general DCP polymatroids. It is not immediately clear that the numbers describing the contraction are small enough to obtain a good characterization, but this task can be worked out. Where can we conclude from this?

Most of our matroids lacks modularity, the approach based on modularity can be used for problems coming form $(k, 0)$-matroids or for the linear cases.

The gain from the other case is that pseudomodular matroid are closed under principal extension, this is good news. We conjecture that (k, l)-matroids are not pseudomodular in general. However, $(k, 0)$-matroids, and $(1, 1)$-matroids satisfying (3.2) are pseudomodular.

The problem of constructing an algorithm for general DCP matroids is therefore an important open question. Clearly, the matroid oracles must be able to choose a non-loop from the intersection of the spans of circuits of a NTCDC, and to be able to do other algorithmic manipulations with flats, but this is a secondary issue. Dress and Lovász [10] remarks that for algebraic matroids we need also the development of oracles testing algebraic independence. Hence, if we are looking for an algorithm for DCP matroids then it should be first developed for finite combinatorial matroids, say solid ones.

The two other algorithms for linear problems, Orlin and Vande Vate's one [45] and Gabow and Stallmann's one [17] heavily rely on linearity we do not know how to extend them to some special non-modular DCP matroids.

For NTCDC-free polymatroids the problem is pretty much solved. A recent result is that Gyula Pap announced the first and only known algorithm for polymatroids without NTCDCs in his doctoral thesis [46].

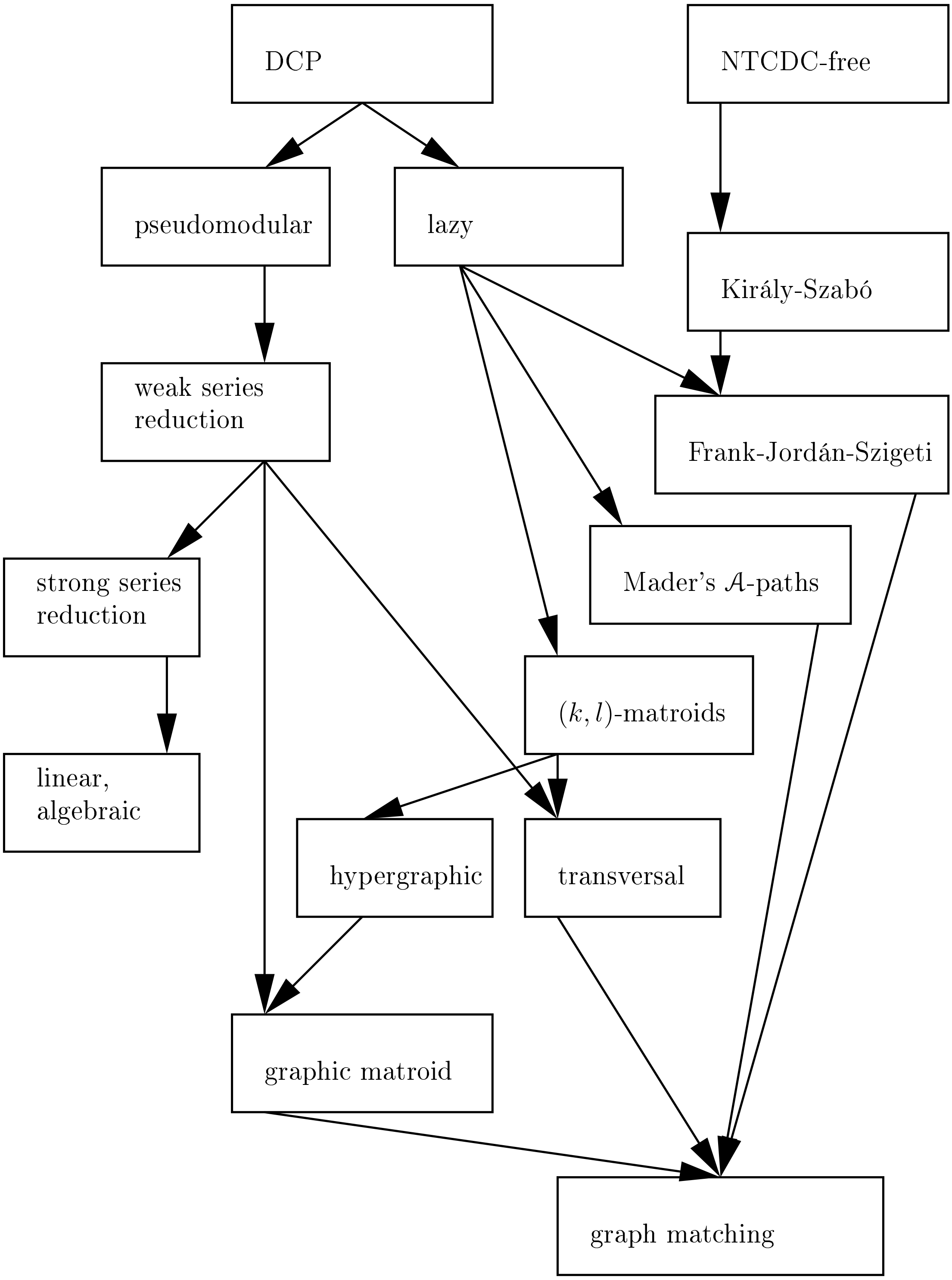

Figure 8.1: Relation of the parity problem of some matroids and polymatroids

Bibliography

[1] Claude Berge. Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris*, 247:258–259, 1958. 4, 26

[2] Piotr Berman, Martin Fürer, and Alexander Zelikovsky. Applications of the linear matroid parity algorithm to approximating Steiner trees. In *Computer science—theory and applications*, volume 3967 of *Lecture Notes in Comput. Sci.*, pages 70–79. Springer, Berlin, 2006. 3

[3] Anders Björner and László Lovász. Pseudomodular lattices and continuous matroids. *Acta Sci. Math. (Szeged)*, 51(3-4):295–308, 1987. 2, 34, 39, 40

[4] André Bouchet. Coverings and delta-coverings. In *Integer programming and combinatorial optimization (Copenhagen, 1995)*, volume 920 of *Lecture Notes in Comput. Sci.*, pages 228–243. Springer, Berlin, 1995. 4

[5] André Bouchet and Bill Jackson. Parity systems and the delta-matroid intersection problem. *Electron. J. Combin.*, 7:Research Paper 14, 22 pp. (electronic), 2000. 4

[6] Gruia Călinescu and Cristina G. Fernandes. Finding large planar subgraphs and large subgraphs of a given genus. In *Computing and combinatorics (Hong Kong, 1996)*, volume 1090 of *Lecture Notes in Comput. Sci.*, pages 152–161. Springer, Berlin, 1996. 3

[7] Gruia Călinescu, Cristina G. Fernandes, Ulrich Finkler, and Howard Karloff. A better approximation algorithm for finding planar subgraphs. In *Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (Atlanta, GA, 1996)*, pages 16–25, New York, 1996. ACM. 3

[8] Gruia Călinescu, Cristina G. Fernandes, Ulrich Finkler, and Howard Karloff. A better approximation algorithm for finding planar subgraphs. *J. Algorithms*, 27(2):269–302, 1998. 7th Annual ACM-SIAM Symposium on Discrete Algorithms (Atlanta, GA, 1996). 3

[9] Robert P. Dilworth. Dependence relations in a semi-modular lattice. *Duke Math. J.*, 11:575–587, 1944. 11, 47

[10] Andreas W. M. Dress and László Lovász. On some combinatorial properties of algebraic matroids. *Combinatorica*, 7(1):39–48, 1987. 2, 5, 15, 27, 33, 38, 88

[11] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 69–87. Gordon and Breach, New York, 1970. 11, 47

[12] Zsolt Fekete. Source location with rigidity and tree packing requirements. Technical Report TR-2005-04, Egerváry Research Group, Budapest, 2005. `www.cs.elte.hu/egres`. 83, 84

[13] András Frank, Tibor Jordán, and Zoltán Szigeti. An orientation theorem with parity conditions. *Discrete Appl. Math.*, 115(1-3):37–47, 2001. 1st Japanese-Hungarian Symposium for Discrete Mathematics and its Applications (Kyoto, 1999). 3, 5, 65, 69

[14] András Frank, Tamás Király, and Zoltán Király. On the orientation of graphs and hypergraphs. *Discrete Appl. Math.*, 131(2):385–400, 2003. Submodularity. 62, 63

[15] András Frank, Éva. Tardos, and András Sebő. Covering directed and odd cuts. *Math. Programming Stud.*, (22):99–112, 1984. Mathematical programming at Oberwolfach, II (Oberwolfach, 1983). 3, 5, 64, 65

[16] Merrick L. Furst, Jonathan L. Gross, and Lyle A. McGeoch. Finding a maximum-genus graph imbedding. *J. Assoc. Comput. Mach.*, 35(3):523–534, 1988. 3

[17] Harold N. Gabow and Matthias F. M. Stallmann. An augmenting path algorithm for linear matroid parity. *Combinatorica*, 6(2):123–150, 1986. Theory of computing (Singer Island, Fla., 1984). 88, 89

[18] Tibor Gallai. Über extreme Punkt- und Kantenmengen. 1959. 16

[19] James F. Geelen and Satoru Iwata. Matroid matching via mixed skew-symmetric matrices. *Combinatorica*, 25(2):187–215, 2005. 4

[20] James F. Geelen, Satoru Iwata, and Kazuo Murota. The linear delta-matroid parity problem. *J. Combin. Theory Ser. B*, 88(2):377–398, 2003. 4

[21] János Geleji. Count matroidok osszegzése. 2007. TDK dolgozat. 42

[22] Thorkell Helgason. Aspects of the theory of hypermatroids. In *Hypergraph Seminar (Proc. First Working Sem., Ohio State Univ., Columbus, Ohio, 1972; dedicated to Arnold Ross)*, pages 191–213. Lecture Notes in Math., Vol. 411. Springer, Berlin, 1974. 8

[23] Winfried Hochstättler and Walter Kern. Matroid matching in pseudomodular lattices. *Combinatorica*, 9(2):145–152, 1989. 2, 34

[24] Bill Jackson and Tibor Jordán. The d-dimensional rigidity matroid of sparse graphs. *J. Combin. Theory Ser. B*, 95(1):118–133, 2005. 45

[25] Thomas A. Jenkyns. Matchoids: A generalization of matchings and matroids, 1974. Ph.D. Thesis, Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario. 1

[26] Per M. Jensen and Bernhard Korte. Complexity of matroid property algorithms. *SIAM J. Comput.*, 11(1):184–190, 1982. 2, 17

[27] Tamás Király and Jácint Szabó. A note on parity constrained orientations. Technical Report TR-2003-11, Egerváry Research Group, Budapest, 2003. `www.cs.elte.hu/egres`. 3, 5, 66, 82

[28] Stephan Kromberg. *Adjoints, Schiefkörper und algebraische Matroide*. PhD thesis, 1995. 34

[29] Gerard Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331–340, 1970. 44

[30] Eugene L. Lawler. Matroids with parity conditions: A new class of combinatorial optimization problems. Memorandum ERL-M334, Electronics Research Laboratory, Berkeley, 1971. 1

[31] Eugene L. Lawler. *Combinatorial optimization: networks and matroids.* Holt, Rinehart and Winston, New York, 1976. 1

[32] László Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages 45–86. Academic Press, London, 1977. 39, 71

[33] László Lovász. Matroid matching and some applications. *J. Combin. Theory Ser. B*, 28(2):208–236, 1980. 1, 2, 3, 5, 15, 44, 55, 66, 75, 83

[34] László Lovász. Selecting independent lines from a family of lines in a space. *Acta Sci. Math. (Szeged)*, 42(1-2):121–131, 1980. 1, 15

[35] László Lovász. The matroid matching problem. In *Algebraic methods in graph theory, Vol. I, II (Szeged, 1978)*, volume 25 of *Colloq. Math. Soc. János Bolyai*, pages 495–517. North-Holland, Amsterdam, 1981. 1, 15, 17, 88

[36] László Lovász. Singular spaces of matrices and their application in combinatorics. *Bol. Soc. Brasil. Mat. (N.S.)*, 20(1):87–99, 1989. 35

[37] László Lovász. The membership problem in jump systems. *J. Combin. Theory Ser. B*, 70(1):45–66, 1997. 4

[38] László Lovász and Michael D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29. 25, 35

[39] László Lovász and Vera T. Sós, editors. *Algebraic methods in graph theory. Vol. I, II*, volume 25 of *Colloquia Mathematica Societatis János Bolyai*. North-Holland Publishing Co., Amsterdam, 1981. Papers from the Conference held in Szeged, August 24–31, 1978. 1, 88

[40] László Lovász and Yechiam Yemini. On generic rigidity in the plane. *SIAM J. Algebraic Discrete Methods*, 3(1):91–98, 1982. 39

[41] Wolfgang Mader. Über die Maximalzahl kreuzungsfreier H-Wege. *Arch. Math. (Basel)*, 31(4):387–402, 1978/79. 5, 55, 56

[42] Márton Makai. Matroid matching with dilworth truncation. *Discrete Mathematics*, 308(8):1394–1404, 2008. 41, 43

[43] Márton Makai and Jácint Szabó. The parity problem of polymatroids without double circuits. *Combinatorica*. accepted. 75, 76

[44] Ladislav Nebeský. A new characterization of the maximum genus of a graph. *Czechoslovak Math. J.*, 31(106)(4):604–613, 1981. 3, 5, 65

[45] James B. Orlin and John H. Vande Vate. Solving the linear matroid parity problem as a sequence of matroid intersection problems. *Math. Programming*, 47(1, (Ser. A)):81–106, 1990. 25, 88, 89

[46] Gyula Pap. A constructive approach to matching and its generalizations. 2006. Ph.D. thesis. 25, 89

[47] Hans Jürgen Prömel and Angelika Steger. RNC-approximation algorithms for the Steiner problem. In *STACS 97 (Lübeck)*, volume 1200 of *Lecture Notes in Comput. Sci.*, pages 559–570. Springer, Berlin, 1997. 3

[48] Hans Jürgen Prömel and Angelika Steger. A new approximation algorithm for the Steiner tree problem with performance ratio 5/3. *J. Algorithms*, 36(1):89–101, 2000. 3

[49] András Recski. *Matroid theory and its applications in electric network theory and in statics*, volume 6 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1989. 25

[50] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Paths, flows, matchings, Chapters 1–38. 6

[51] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69. 6, 11, 29, 43, 47, 88

[52] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. C*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Disjoint paths, hypergraphs, Chapters 70–83. 6, 56

[53] Jacob T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *J. Assoc. Comput. Mach.*, 27(4):701–717, 1980. 35

[54] Leonard Tan. The weak series reduction property implies pseudomodularity. *European J. Combin.*, 18(6):713–719, 1997. 34

[55] Po Tong, Eugene L. Lawler, and Vijay V. Vazirani. Solving the weighted parity problem for gammoids by reduction to graphic matching. In *Progress in combinatorial optimization (Waterloo, Ont., 1982)*, pages 363–374. Academic Press, Toronto, ON, 1984. 5, 43

[56] John H. Vande Vate. Structural properties of matroid matchings. *Discrete Appl. Math.*, 39(1):69–85, 1992. 25

[57] Neil White and Walter Whiteley. A class of matroids defined on graphs and hypergraphs by counting properties. 1984. preprint. 4

[58] Walter Whiteley. Some matroids from discrete applied geometry. In *Matroid theory (Seattle, WA, 1995)*, volume 197 of *Contemp. Math.*, pages 171–311. Amer. Math. Soc., Providence, RI, 1996. 4

Summary

The dissertation focuses on matching problems of polymatroids arising in combinatorial and graph theoretical applications. The problem is computationally hard in general, and all the previously known tractable cases are tractable due to the double circuit property (DCP). In the first half we present new polymatroid classes having the DCP, and also applications. The polymatroids of the second half need a completely different approach. Here we shown that parity problems of polymatroids without non-trivial compatible double circuits (NTCDCs) also can be handled.

First we deal with (k,l)-matroids which are natural generalizations of transversal and graphic matroids. We prove that if the hypergraph (the ground set of the matroid) satisfies a certain density condition, then the matroid has the DCP. Then, we extend (k,l)-matroids by the class of solid polymatroids. Solid polymatroids are defined by intersecting submodular functions, and the defining function must satisfy some simple abstract properties to have the DCP. As an application we get the formula for Mader's vertex-disjoint $\mathcal{A}$-path problem, by proving that the polymatroid in Lovász' formulation has a solid embedding.

Parity constrained connectivity problems have a central role in the dissertation. We show that Frank, Jordán, and Szigeti's theorem which characterizes the graphs with rooted k-edge-connected parity constrained orientations is an immediate consequence of the parity of solid polymatroids. Very little is known about parity constrained orientations where the connectivity requirement is given by a crossing supermodular function, specially it is open to characterize the graphs having strongly connected orientations with even out-degrees. We solve the problem for the planar case, again by solid polymatroids.

Király and Szabó extended Frank, Jordán, and Szigeti's theorem to the case when the connectivity requirement is given by an intersecting supermodular function. This problem is unlikely to reduce to the parity of polymatroids having the DCP. We have shown that these polymatroids have no NTCDCs, and that a partition type formula characterizes the matchings of polymatroids without NTCDCs. Other interesting applications are a pinning down and a connectivity augmentation problem of Fekete.

Összefoglaló

A doktori disszertációban kombinatorikai és gráfelméleti alkalmazásokban felmerülő polimatroidok párosítási feladataival foglalkozunk. A feladat bonyolultságelméleti szempontból nehéz, s az összes eddig ismert kezelhető eset kezelhetősége a dupla kör tulajdonságnak (DCP) köszönhető. A dolgozat első felében új DCP tulajdonságú polimatroidokat konstruálunk alkalmazásokkal. A második rész polimatroidjai teljesen más megközelítést igényelnek, itt megmutatjuk, hogy a nemtriviális kompatibilis dupla köröket (NTCDC) nem tartalmazó polimatroidok párosítási feladata is kezelhető. Célunk a kombinatorikus optimalizálás minél több polimatroid párosítási feladatának közös keretbe foglalása.

Először (k, l)-matroidokkal foglalkozunk, melyek transzverzális és körmatroidok természetes általánosításai. Megmutatjuk, hogy ha a hipergáf (a matroid alaphalmaza) elég sűrű, akkor a matroid DCP. A (k, l)-matroidokat a tömör polimatroidok osztályával általánosítjuk. Tömör polimatroidokat metsző szubmoduláris függvénnyel lehet definiálni, s a DCP érdekében a definiáló függvénynek teljesíteni kell bizonyos abszrakt tulajdonságokat. Mader pontdiszjunkt $\mathcal{A}$-utas feladata ennek érdekes alkalmazása, hisz mutatunk a Lovász által adott párosítási megfogalmazásában szereplő polimatroid egy egyszerű tömör beágyazását.

A paritásos összefüggőségi irányítási feladatok központi szerepet játszanak a dolgozatban. Megmutatjuk, hogy Frank, Jordán, és Szigeti tétele ami karakterizálja a paritásos gyökeresen k-élösszefüggő irányítással rendelkező gráfokat tömör polimatroidok párosításából levezethető. Nagyon keveset tudunk olyan paritásos irányítási feladatokról, melyekben az összefüggőségi igényt keresztező szupermoduláris függvény írja le, speciálisan nyitott a csupa páros kifokú erősen összefüggő irányítással rendelkező gráfok karakterizációja. Megmutatjuk, hogy síkgráfok esetén ez utóbbi feladat felírható egy tömör polimatroid párosításaként.

Király és Szabó általánosították Frank, Jordán, és Szigeti tételét arra az esetre amikor az összefüggéségi igényt metsző szupermoduláris függvény adja. Valószínűtlen, hogy ez az általános feladat DCP tulajdonságú polimatroid párosítására vezetne. Megmutatjuk, hogy ezen feladat felírható NTCDC mentes polimatroid párosításaként, s hogy ilyen polimatroidokra egy partíciós formula karakterizálja a legnagyobb párosítás méretét. Két további alkalmazás Fekete egy síkbeli leszúrási s egy összefüggőségnövelési tétele.